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R. C. Thorne

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# THE ASYMPTOTIC EXPANSION OF LEGENDRE FUNCTIONS OF LARGE DEGREE AND ORDER†

By R. C. THORNE

*Trinity College, University of Cambridge, and The California Institute of Technology, Pasadena, California*

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CONTENTS



New expansions for the Legendre functions $P_n^{-m}(z)$ and $Q_n^{-m}(z)$ are obtained; $m$ and $n$ are large positive numbers, $0<m<n$ and $\alpha=m/(n+\frac{1}{2})$ is kept fixed as $n\to\infty$; $z$ is an unrestricted complex variable. Three groups of expansions are obtained. The first is in terms of exponential functions. These expansions are uniformly valid as $n\to\infty$ with respect to $z$ for all $z$ lying in $\mathscr{R}z\geqslant 0$ except for the strips given by $|\mathscr{I}z|<\delta$, $\mathscr{R}z<\beta+\delta$, where $\delta>0$ and $\beta=\sqrt{(1-\alpha^2)}$. The second set of expansions is in terms of Airy functions. These expansions are uniformly valid with respect to $z$ throughout the whole $z$ plane cut from $+1$ to $-\infty$ except for a pear-shaped domain surrounding the point $z=-1$ and a strip lying immediately below the real $z$ axis for which $|\mathscr{R}z|<\beta+\delta$, $0\geqslant\mathscr{I}z>-\delta$. The third group of expansions is in terms of Bessel functions of order $m$. These expansions are valid uniformly with respect to $z$ over the whole cut $z$ plane except for the pear-shaped domain surrounding $z=-1$. No expansions have been given before for the Legendre functions of large degree and order.

## 1. INTRODUCTION AND SUMMARY

This paper is concerned with the investigation of the solutions of the differential equation

$$(1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+\left\{n(n+1)-\frac{m^2}{1-z^2}\right\}w=0, \qquad (1\cdot1)$$

in the case when $m$ and $n$, with $0<m<n$, are large positive not necessarily integral numbers and $z$ is an unrestricted complex variable. The equation (1·1), known as Legendre's equation, occurs in potential theory and in other branches of applied mathematics; a knowledge of the behaviour of its solutions for large values of its defining parameters is therefore of interest.

We take as the fundamental system of solutions of (1·1) the functions $P_n^{-m}(z)$ and $Q_n^{-m}(z)$, known respectively as the Legendre functions of the first and second kind of degree $n$ and

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order $m$ (MacRobert 1947, p. 122). Hobson's definitions of these functions will be used (see §2). Then $P_n^{\pm m}(z)$ and $Q_n^{\pm m}(z)$ are single-valued analytic functions in the $z$ plane cut along the real axis from $+1$ to $-\infty$, and are real when $z$ is real and $z>1$. Except where otherwise stated we take $|\arg z|\leqslant\pi$, $|\arg(z\pm 1)|\leqslant\pi$. The equation (1·1) has regular singularities at $z=1$, $z=-1$, $|z|=\infty$. When $0<m<n$, $P_n^{-m}(z)$ is the only solution of (1·1) which is bounded at $z=1$ (but see (2·1)); $Q_n^{-m}(z)$ is bounded at infinity. Thus in this paper detailed discussion is restricted to the functions $P_n^{-m}(z)$ and $Q_n^{-m}(z)$. Properties of $P_n^{m}(z)$ and $Q_n^{m}(z)$ can be easily deduced using the connecting formulae (2·12) and (2·13). For $z=x$ where $-1<x<1$, the fundamental solutions of (1·1) are taken as $\mathrm{P}_n^{-m}(x)$ and $\mathrm{Q}_n^{-m}(x)$ defined in (2·2) and (2·3); these functions are described as Ferrers's functions (but see §2), and are used in many branches of applied mathematics. They are real for these values of $x$. Normalizing (1·1) we conclude that the differential equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2}=\left\{\frac{n(n+1)}{z^2-1}+\frac{m^2-1}{(z^2-1)^2}\right\}w, \tag{1·2}$$

has the solutions $(z^2-1)^{\frac{1}{2}}P_n^{-m}(z)$, $(z^2-1)^{\frac{1}{2}}Q_n^{-m}(z)$.

In this paper asymptotic expansions for the functions $P_n^{-m}(z)$ and $Q_n^{-m}(z)$ (and thence for $\mathrm{P}_n^{-m}(x)$ and $\mathrm{Q}_n^{-m}(x)$) are derived. The expansions are valid uniformly with respect to $z$ throughout certain domains in the cut $z$ plane as $m\to\infty$ and $n\to\infty$ where $0<m<n$ and $0<\alpha<1$, $\alpha=m/(n+\frac{1}{2})$ and $\alpha$ is kept fixed; the author surmises that the expansions are probably valid for those values of $m$ for which $An^{-\frac{1}{3}}<\alpha<1-Bn^{-\frac{1}{3}}$; $A>0$, $B>0$. The method adopted to obtain the expansions is that developed by Olver (1954*b*) and is based upon the consideration of equation (1·2), using the well-known idea that approximately identical differential equations have approximately identical solutions. In a recent paper (Thorne 1957*a*, hereafter referred to as I) the conditions under which Olver's theory can be applied to (1·2) when $m$ and $n$ satisfy the above conditions were investigated (I, §5). It was shown that if we set $u=n+\frac{1}{2}$, $m=\alpha u$, $\alpha$ fixed as $u\to\infty$ and $0<\alpha<1$, so that (1·2) becomes

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2}=\left\{\frac{z^2-\beta^2}{(z^2-1)^2}u^2-\frac{z^2+3}{4(z^2-1)^2}\right\}w, \tag{1·3}$$

where $\beta=\surd(1-\alpha^2)$, it is possible to obtain asymptotic expansions for $P_n^{-m}(z)$ and $Q_n^{-m}(z)$ which are valid uniformly with respect to $z$, as $u\to\infty$, for $z$ lying in a domain $\mathbf{D}_z$, say, in which the points $z=1$, $z=\beta+\mathrm{i}0$ are interior points and which extends to infinity. These expansions are in terms of Airy functions. The points $z=-\beta$, $z=-1$, $z=\beta-\mathrm{i}0$ do not lie in $\mathbf{D}_z$ and the expansions are not valid in the strip $0\geqslant\mathscr{I}z>-\delta$, $\mathscr{R}z<\beta+\delta$ $(\delta>0)$. These Airy-type expansions form the second of the three groups of expansions derived in this paper. We shall now discuss some of the properties of these expansions.

The form (1·3) of the equation (1·1) is relevant to each of these three groups of expansions. The coefficient of $u^2w$ in (1·3) has simple zeros at $z=\pm\beta$, and these points are known as turning points of (1·3). The equation also has regular singularities at $z=\pm 1$, $|z|=\infty$. The turning points and singularities are significant in the determination of the asymptotic character of the solutions of a differential equation.

First, in §3, we restrict the consideration of equation (1·3) to the domain $\mathbf{Z}''_z$, say, consisting of the half-plane $\mathscr{R}z\geqslant 0$ from which have been removed the strips $|\mathscr{I}z|<\delta$, $\mathscr{R}z<\beta+\delta$ $(\delta>0)$. In $\mathbf{Z}''_z$ (1·3) has two regular singularities and no turning points. Now the equation satisfied by exp $(\pm u\xi)$ has no turning points in the $\xi$ plane, and has an irregular singularity at infinity. From the theory developed by Olver (1954*b*, theorem A), we then deduce in § 3 that the Legendre functions have asymptotic expansions, as $u \to \infty$, in terms of the exponential functions exp $(\pm u\xi)$ where $\xi \equiv \xi(z)$. These expansions are valid uniformly with respect to $z$ in $\mathbf{Z}''_z$ (and sometimes in larger domains), but are never valid at the turning point $z = \beta$. These exponential-type expansions form the first of the three groups of expansions of this paper.

It is not possible to obtain asymptotic expansions which are valid at a turning point and are in terms of exponential functions, but Olver (1954*b*), following earlier writers, has shown that under these circumstances it is convenient to use Airy functions. These functions satisfy the differential equation

$$\frac{d^2W}{d\zeta^2} = u^2\zeta W, \tag{1·4}$$

which has a single turning point at $\zeta = 0$. Now the functions $P_n^{-m}(z)$ and $Q_n^{-m}(z)$ take on different values at the two points $z = \beta + i0$ and $z = \beta - i0$, and so these points are effectively two distinct turning points of the Legendre functions, and since (1·4) has only one turning point, we see why the Airy-type expansions mentioned above are valid at $z = \beta + i0$ but are not valid at $z = \beta - i0$. The existence of the Airy-type expansions was proved in I, § 5, but no attempt was made to derive them. They are obtained in § 4 of this paper.

When there are two turning points for a differential equation asymptotic expansions in terms of Airy functions for the solutions of the equation will not be valid at both points, but in the preceding paper (Thorne 1957*b*) it was shown that under certain circumstances, when the two points are separated symmetrically by a regular singularity of a particular kind, it is possible to obtain expansions valid at the two turning points and at the singularity. These expansions are in terms of the Bessel functions $I_m(ut)$, $K_m(ut)$ and $t^{\frac{1}{2}}I_m(ut)$, $t^{\frac{1}{2}}K_m(ut)$ are solutions of the differential equation

$$\frac{d^2y}{dt^2} = \left\{u^2\left(1 + \frac{\alpha^2}{t^2}\right) - \frac{1}{4t^2}\right\}y, \tag{1·5}$$

where $m = \alpha u$. The equation (1·5) has two turning points at $t = \pm i\alpha$, a regular singularity at $t = 0$ and an irregular singularity at infinity. We show in § 5 that (1·5) is a suitable equation for comparison with (1·3) and the points $z = \beta \pm i0$, $z = 1$ correspond to the points $t = \pm i\alpha$, $t = 0$ respectively. We then deduce that there exist asymptotic expansions for the Legendre functions in terms of the Bessel functions which are uniformly valid, as $u \to \infty$, in a domain which extends to infinity and in which $z = 1$, $z = \beta \pm i0$ are interior points. These expansions are thus valid in a domain larger than that for the Airy-type expansions of § 4. The Bessel-type expansions form the last of the three groups of expansions in this paper.

Except for a single-term approximation of Jeffreys, discussed in § 3, no expansions have been given previously for the Legendre functions of large degree and order. However, expansions for which the order $m$ is fixed and the degree $n$ is large have been given before; these are discussed in § 6; they are in terms of Bessel functions of fixed order.

*Summary.* Relevant properties of the Legendre functions are given in § 2. In § 3 Olver's theory (1954*b*, § 5) is applied to obtain elementary asymptotic expansions for the Legendre functions. These are in terms of exponential functions and are given in (3·18) and (3·19). Expansions for the derivatives of the functions are given in (3·32) and (3·33), and expansions for the Ferrers functions in (3·28) to (3·31) and (3·35) to (3·38). The Airy-type expansions are derived in § 4, and are given in (4·19), (4·20), (4·22), (4·23) and (4·26). Finally, in § 5 Bessel-type expansions for $P_n^{-m}(z)$, $Q_n^{-m}(z)$ are developed. These are given in (5·17) to (5·21). Previous results and concluding remarks are given in § 6. Throughout this present paper, the preceding paper (Thorne 1957*b*) is referred to as II.

## 2. Relevant properties of the Legendre functions

Hobson's definitions of the Legendre functions $P_n^{-m}(z)$, $Q_n^{-m}(z)$ are used (Hobson 1931, pp. 188, 195). These definitions are in terms of contour integrals and are valid for unrestricted values of $m$ and $n$. When $m$ is a positive integer and $z$ lies in the cut plane we have (Hobson 1931, pp. 187, 194, 205),

$$\left.\begin{aligned} P_n^m(z) = (z^2-1)^{\frac{1}{2}m}\frac{\mathrm{d}^m P_n(z)}{\mathrm{d}z^m}, \quad Q_n^m(z) = (z^2-1)^{\frac{1}{2}m}\frac{\mathrm{d}^m Q_n(z)}{\mathrm{d}z^m}, \\ P_n^{-m}(z) = \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}P_n^m(z). \end{aligned}\right\} \tag{2·1}$$

When $m$ is not an integer $P_n^{-m}(z)$ and $P_n^m(z)$ are linearly independent (see (2·7)), but $Q_n^{-m}(z)$ is a linear multiple of $Q_n^m(z)$ for all $m$ (see (2·10)).

For $z = x$, $-1<x<1$, the fundamental solutions of (1·1) are taken as $\mathrm{P}_n^{-m}(x)$ and $\mathrm{Q}_n^{-m}(x)$ defined by the relations

$$\mathrm{P}_n^{-m}(x) = \mathrm{e}^{\mp\frac{1}{2}m\pi\mathrm{i}}\,P_n^{-m}(x\pm\mathrm{i}0), \tag{2·2}$$

$$2\mathrm{e}^{-m\pi\mathrm{i}}\,\mathrm{Q}_n^{-m}(x) = \mathrm{e}^{\frac{1}{2}m\pi\mathrm{i}}\,Q_n^{-m}(x+\mathrm{i}0)+\mathrm{e}^{-\frac{1}{2}m\pi\mathrm{i}}\,Q_n^{-m}(x-\mathrm{i}0), \tag{2·3}$$

where $f(x\pm\mathrm{i}0) = \lim_{\epsilon\to 0} f(x\pm\mathrm{i}\epsilon)$, $\epsilon>0$. Also

$$-\mathrm{i}\pi\,\mathrm{e}^{-m\pi\mathrm{i}}\,\mathrm{P}_n^{-m}(x) = \mathrm{e}^{\frac{1}{2}m\pi\mathrm{i}}\,Q_n^{-m}(x+\mathrm{i}0)-\mathrm{e}^{-\frac{1}{2}m\pi\mathrm{i}}\,Q_n^{-m}(x-\mathrm{i}0). \tag{2·4}$$

The expressions (2·2), (2·3) and (2·4) are given in Hobson (1931, pp. 227, 229 and 228 respectively). From (2·3) and (2·4) we deduce that

$$\mathrm{Q}_n^{-m}(x) = \mathrm{e}^{\frac{3}{2}m\pi\mathrm{i}}\,Q_n^{-m}(x+\mathrm{i}0)+\tfrac{1}{2}\mathrm{i}\pi\mathrm{P}_n^{-m}(x). \tag{2·5}$$

When $m$ is a positive integer

$$\left.\begin{aligned} \mathrm{P}_n^m(x) = (-1)^m(1-x^2)^{\frac{1}{2}m}\frac{\mathrm{d}^m P_n(x)}{\mathrm{d}x^m} = (-1)^m\,T_n^m(x), \\ \mathrm{Q}_n^m(x) = (-1)^m(1-x^2)^{\frac{1}{2}m}\frac{\mathrm{d}^m Q_n(x)}{\mathrm{d}x^m} = (-1)^m\,U_n^m(x). \end{aligned}\right\} \tag{2·6}$$

Both pairs of functions $\mathrm{P}_n^m$, $\mathrm{Q}_n^m$ and $T_n^m$, $U_n^m$ have been described as Ferrers's functions (see MacRobert 1947, p. 307; Erdélyi 1953, p. 179 and Whittaker & Watson 1920, p. 323); we shall be dealing only with the functions $\mathrm{P}_n^{-m}$ and $\mathrm{Q}_n^{-m}$ in this paper. For the Ferrers functions we follow the notation introduced in Erdélyi (1953, chapter 3); this avoids the confusion resulting from Hobson's use of the same symbol to denote $P_n^{-m}$ and $\mathrm{P}_n^{-m}$. The Ferrers functions, defined originally when $z = x$ is real and $|x|<1$, can be continued analytically on to the $z$ plane so that $\mathrm{P}_n^{\pm m}(z)$ and $\mathrm{Q}_n^{\pm m}(z)$ are single-valued analytic functions of $z$ on the whole $z$ plane cut, in this case, along the real axis from $+1$ to $+\infty$ and from $-1$ to $-\infty$. They are real when $z$ is real and $|z|<1$.

For $|z-1|<2$

$$P_n^{-m}(z) = \frac{1}{\Gamma(m+1)}\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}m} F(-n, n+1; m+1; \tfrac{1}{2}-\tfrac{1}{2}z), \tag{2·7}$$

and for $|z|>1$

$$e^{m\pi i} Q_n^{-m}(z) = 2^n \frac{\Gamma(n+1)\,\Gamma(n-m+1)}{\Gamma(2n+2)} \frac{(z^2-1)^{-\frac{1}{2}m}}{z^{n-m+1}} F\left(\tfrac{1}{2}n-\tfrac{1}{2}m+1, \tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2}; n+\tfrac{3}{2}; \frac{1}{z^2}\right). \tag{2·8}$$

For all values of $z$

$$P_n^{-m}(z) = \frac{e^{m\pi i}}{\pi \cos n\pi}\{\sin(n-m)\,\pi Q_n^{-m}(z) - \sin(n+m)\,\pi Q_{-n-1}^{-m}(z)\}, \tag{2·9}$$

$$Q_n^{-m}(z) = \frac{\pi\, e^{-m\pi i}}{2\sin m\pi}\left\{\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} P_n^m(z) - P_n^{-m}(z)\right\} = e^{-2m\pi i}\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} Q_n^m(z). \tag{2·10}$$

The formulae (2·7) to (2·10) can be found in Hobson (1931, pp. 188, 195, 204, 204, 196 respectively). Finally, we have as $z \to 0+i0$, from Erdélyi (1953, 3·2 (40)),

$$Q_n^{-m}(z) = \frac{\sqrt{\pi}\, e^{-\frac{1}{2}(n+m+1)\pi i}\,\Gamma(\frac{1}{2}n-\frac{1}{2}m+\frac{1}{2})}{2^{m+1}\Gamma(\frac{1}{2}n+\frac{1}{2}m+1)} + O(z). \tag{2·11}$$

*Continuation formulae*

For all $z$ there exist the continuation formulae (Hobson 1931, p. 207),

$$P_n^{-m}(-z) = e^{\mp n\pi i} P_n^{-m}(z) - \frac{2}{\pi}\sin(n-m)\,\pi\, e^{m\pi i} Q_n^{-m}(z), \tag{2·12}$$

$$Q_n^{-m}(-z) = -e^{\pm n\pi i} Q_n^{-m}(z), \tag{2·13}$$

where the upper and lower signs are taken according as $\mathscr{I}z>0$ or $\mathscr{I}z<0$. These expressions can easily be extended. Let $z$ circulate the origin and cross the cut $-\infty<z<-1$ but not cross the cut $-1<z<1$. Then from (2·8) and (2·9) we deduce that

$$\left.\begin{aligned} Q_n^{-m}(z\,e^{r\pi i}) &= e^{-r(n+1)\pi i} Q_n^{-m}(z), \\ P_n^{-m}(z\,e^{r\pi i}) &= e^{rn\pi i} P_n^{-m}(z) - \frac{2i\sin(n-m)\,\pi}{\pi\cos n\pi}\sin r(n+\tfrac{1}{2})\,\pi\, e^{(m-\frac{1}{2}r)\pi i} Q_n^{-m}(z), \end{aligned}\right\} \tag{2·14}$$

where $r$ is an arbitrary integer.

If $z$ circulates around the point $z=+1$ and does not cross the cut $-\infty<z<-1$, we can write $z_s = 1+(z-1)\,e^{s\pi i}$, where $|\arg(z-1)|<\pi$ and $s$ is an arbitrary integer. Then from (2·7) and (2·10) it follows that

$$\left.\begin{aligned} P_n^{-m}(z_s) &= e^{sm\pi i} P_n^{-m}(z), \\ Q_n^{-m}(z_s) &= e^{-sm\pi i} Q_n^{-m}(z) - i\pi\, e^{-m\pi i}\sin sm\pi \operatorname{cosec} m\pi P_n^{-m}(z). \end{aligned}\right\} \tag{2·15}$$

The function $Q_n^m(z, 1-)$ defined in Erdélyi (1953, p. 142) corresponds to (2·15*b*) with $s=-1$.

*Asymptotic formulae for gamma functions*

Stirling's series gives, for large values of $u$,

$$\ln\left\{\frac{e^u\,\Gamma(1+u)}{u^u\sqrt{(2\pi u)}}\right\}\sim R_u, \tag{2·16}$$

where $R_u$ is a series consisting solely of terms of odd positive powers of $u^{-1}$. Also for large values of $z$,

$$\ln\{\Gamma(z+\tfrac{1}{2})\}\sim\tfrac{1}{2}\ln 2\pi+z\ln z-z-\frac{1}{2.12z}+\frac{7}{8.360z^2}-\dots,$$

and hence if $u = n+\frac{1}{2}$ and $m = \alpha u$, it follows that for large values of $u$,

$$\ln\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}\sim-2m\ln\frac{2m}{ep}-\frac{\alpha}{12\beta^2u}+\frac{7\alpha(\alpha^2+3)}{4.360\beta^6u^3}-\dots$$

$$\sim-2m\ln\frac{2m}{ep}+T_u, \tag{2·17}$$

where $p = \gamma^\gamma(1+\gamma)^{-1-\gamma}$, $\gamma = \frac{1}{2}(\alpha^{-1}-1)$ and $T_u$ consists only of odd positive powers of $u^{-1}$. Finally,

$$\Gamma(z)\,\Gamma(1-z) = \pi\operatorname{cosec} z\pi, \quad 2^{2z-1}\Gamma(z)\,\Gamma(z+\tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2z). \tag{2·18}$$

## 3. Expansions for the Legendre functions in terms of exponential functions

In this section we derive asymptotic expansions for the Legendre functions which are valid uniformly with respect to $z$ in a domain which extends to infinity and in which $z = 1$ is an interior point; the expansions are in terms of exponential functions. The functions $(z^2-1)^{\frac{1}{2}}P_n^m(z)$, $(z^2-1)^{\frac{1}{2}}Q_n^m(z)$ are solutions of the differential equation (1·3). The coefficient of $u^2w$ in (1·3) has double poles at $z=\pm 1$, and simple zeros at $z=\pm\beta$; the equation has a regular singularity at infinity. Let $\mathbf{Z}_z$ be the half-plane $|\arg z|\leqslant\frac{1}{2}\pi$, cut from $z=0$ to $z=1$, and let $\mathbf{Z}'_z=\mathbf{Z}_z-\mathbf{\Delta}_z$, where $\mathbf{\Delta}_z$ is a small yet finite region surrounding $z=\beta$ consisting of two parts $\mathbf{\Delta}_{z+}$ and $\mathbf{\Delta}_{z-}$ on the upper and lower side of the cut on the real $z$ axis. Then the coefficient of $u^2w$ in (1·3) has a double pole in $\mathbf{Z}'_z$ at $z=1$, and is non-zero, except at infinity, elsewhere in $\mathbf{Z}'_z$. Thus the equation (1·3) considered in the domain $\mathbf{Z}'_z$ is a particular example of a type of equation, described as case C, which has been investigated by Olver (1954*b*, pp. 309, 313). Quoting the result contained in his theorem A (Olver 1954*b*, § 5), we deduce that there exist expansions for solutions of (1·3), as $u\to\infty$, valid in subdomains of $\mathbf{Z}'_z$ in which $z=1$ is an interior point. We shall now obtain these expansions and show that they are also valid for unbounded values of $|z|$ in $\mathbf{Z}'_z$. Introducing new variables $W$, $\xi$ (Olver 1954*b*, (2·4)) by the relations

$$\left(\frac{d\xi}{dz}\right)^2=\frac{z^2-\beta^2}{(z^2-1)^2}, \quad W=\left(\frac{dz}{d\xi}\right)^{-\frac{1}{2}}w=\frac{(z^2-\beta^2)^{\frac{1}{4}}}{(z^2-1)^{\frac{1}{2}}}w, \tag{3·1}$$

we find that $W(\xi)$ satisfies the equation

$$\frac{d^2W}{d\xi^2}=\{u^2+f(\xi)\}\,W, \tag{3·2}$$

where

$$f(\xi)=-\tfrac{1}{2}\{z,\xi\}-\frac{z^2+3}{4(z^2-1)^2}\left(\frac{dz}{d\xi}\right)^2 \tag{3·3}$$

(see II, (2·2)).

Under the $z$-$\xi$ transformation the domain $\mathbf{Z}_z$ becomes a domain $\mathbf{Z}_\xi$, say, in the $\xi$ plane. We choose the positive sign in taking the square root in the first part of (3·1), and an integration constant is added so that $\mathbf{Z}_\xi$ is symmetrical about the real $\xi$ axis. As in previous papers we use subscripts for domains and points to indicate the particular complex plane under consideration; thus $\mathbf{Z}_z$ and $\mathbf{Z}_\xi$ refer to corresponding domains in the $z$ and $\xi$ planes. From (3·1) we deduce that

$$\xi = -\rho + \tfrac{1}{2}\mathrm{i}\alpha\pi,$$

where

$$\rho = -\int_\beta^z \frac{\sqrt{(z^2-\beta^2)}}{z^2-1}\,\mathrm{d}z = \alpha\cosh^{-1}\left\{\frac{\alpha z}{\beta\sqrt{(1-z^2)}}\right\} - \cosh^{-1}\frac{z}{\beta}, \tag{3·4}$$

$$= \alpha\tanh^{-1}\left\{\frac{\sqrt{(z^2-\beta^2)}}{\alpha z}\right\} - \tanh^{-1}\left\{\frac{\sqrt{(z^2-\beta^2)}}{z}\right\}; \tag{3·5}$$

$$f(\xi) = \frac{z^2-1}{4(z^2-\beta^2)^3}\{z^2(4\alpha^2-1)+(1-\alpha^4)\}. \tag{3·6}$$

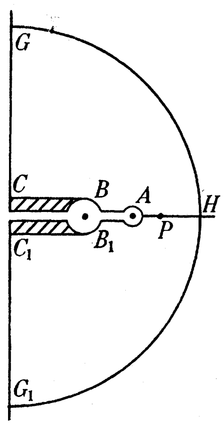


FIGURE 1. $z$ plane.

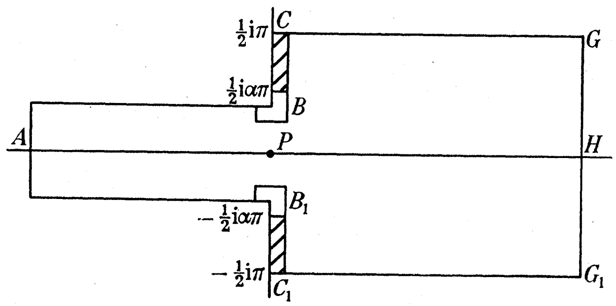


FIGURE 2. $\xi$ plane.

In (3·4) and (3·5) we specify that for $z$ real and $z>1$, $\arg(1+z) = \arg z = 0$ and $\arg(z^{-2}-1) = -\pi$. For $z$ elsewhere on the cut plane appropriate values of the arguments are given by continuity from these values. The $z$-$\rho$ transformation is discussed in more detail in § 4 and we state the following results here. The domain $\mathbf{Z}_\xi$ consists of $\mathscr{R}\xi \leqslant 0$, $|\mathscr{I}\xi| \leqslant \tfrac{1}{2}\alpha\pi$, together with $\mathscr{R}\xi \geqslant 0$, $|\mathscr{I}\xi| \leqslant \tfrac{1}{2}\pi$. Then $\mathbf{Z}'_\xi = \mathbf{Z}_\xi - \mathbf{\Delta}_\xi$, where $\mathbf{\Delta}_\xi$ corresponds to $\mathbf{\Delta}_z$ and consists of $\mathbf{\Delta}_{\xi+}$ and $\mathbf{\Delta}_{\xi-}$ centred respectively on $\xi = \pm\tfrac{1}{2}\mathrm{i}\alpha\pi$; for simplicity we suppose that $\mathbf{\Delta}_{\xi+}$ and $\mathbf{\Delta}_{\xi-}$ are portions of squares, centred respectively on $\tfrac{1}{2}\mathrm{i}\alpha\pi$ and $-\tfrac{1}{2}\mathrm{i}\alpha\pi$, having sides of length $2\delta > 0$, where $\delta < \min\{\tfrac{1}{2}\alpha\pi, \tfrac{1}{2}(1-\alpha)\pi\}$. Let $\mathbf{R}_\xi$ denote the two strips $0 \leqslant \mathscr{R}\xi < \delta$, $\tfrac{1}{2}\alpha\pi + \delta < |\mathscr{I}\xi| < \tfrac{1}{2}\pi$ and let $\mathbf{Z}''_\xi = \mathbf{Z}'_\xi - \mathbf{R}_\xi$. These domains are shown in figures 1 and 2.

Using the logarithmic form of (3·5), we deduce that as $|z| \to \infty$

$$\mathscr{R}\xi \to \infty, \quad z \sim \tfrac{1}{2} r\beta\,\mathrm{e}^{\xi}, \quad \text{where} \quad \ln r = \alpha \ln \beta^{-1}(1+\alpha), \tag{3·7}$$

and as $|z| \to 1$

$$\mathscr{R}\xi \to -\infty, \quad (z-1) \sim \tfrac{1}{2}p^2 \exp(2\alpha^{-1}\xi), \quad p \text{ as in (2·15)}. \tag{3·8}$$

Thus we conclude from (3·6) that $f(\xi)$ is a regular function of $\xi$ throughout $\mathbf{Z}'_\xi$ and $f(\xi) = O(|\xi|^{-2})$ as $|\xi| \to \infty$ in $\mathbf{Z}'_\xi$. Thus $f(\xi)$ satisfies in $\mathbf{Z}'_\xi$ all the preliminary conditions for the application of theorem A of Olver (1954*b*) concerning the asymptotic solutions of equation (3·2). It is not difficult to show that if we had chosen any other larger parameter $u_1$, say, where $u_1^2 = u^2 - r$, where $r \neq 0$, the function $f_1(\xi)$, say, corresponding to $f(\xi)$ in (3·2) would be such that $f_1(\xi) = \text{constant} + O(|\xi|^{-2})$ as $|\xi|\to\infty$ in $\mathbf{Z}'_\xi$; this condition is not sufficient for the application of theorem A when $|\xi|\to\infty$ in $\mathbf{Z}'_\xi$. We can also deduce that our choice of $u$ is the correct one from an examination of (4·3), I, § 5, and Olver (1956, appendix).

If we wish we can take $\mathbf{Z}'_\xi$ to be the domain $\mathbf{D}$ of theorem A of Olver (1954*b*), and it would then be necessary to define a domain $\mathbf{D}'$, lying wholly within $\mathbf{Z}'_\xi$, the boundary of $\mathbf{D}'$ consisting of lines drawn parallel to the boundaries of $\mathbf{Z}'_\xi$, and at a small distant $\delta'$ ($<\frac{1}{2}\delta$, say) from them. Expansions for $W(\xi)$ of (3·2) would then be derived for $\xi$ in $\mathbf{D}'$. But $\mathbf{D}'$ can be taken, in fact, to be the domain $\mathbf{Z}'_\xi$ itself, since $f(\xi)$ satisfies all the conditions of theorem A in a domain formed from $\mathbf{Z}'_\xi$ by 'moving back', by an amount $\delta'$ ($<\frac{1}{2}\delta$), all the boundary lines of $\mathbf{Z}'_\xi$. Thus we take $\mathbf{Z}'_\xi$ to be the domain $\mathbf{D}'$ of theorem A. We take the points $a_1$ and $a_2$ of this theorem to be the points at infinity on the negative and positive real $\xi$-axes, respectively. The domains $\mathbf{D}_1$ and $\mathbf{D}_2$ of theorem A are then $\mathbf{Z}''_\xi$ and $\mathbf{Z}'_\xi$ respectively.

Thus if functions $T_s(\xi)$ are defined by the relations

$$T_0(\xi) = 1, \quad T_{s+1}(\xi) = -\tfrac{1}{2}T'_s(\xi) + \frac{1}{2}\int^\xi f(\xi)\, T_s(\xi)\, \mathrm{d}\xi \quad (s \geqslant 0), \tag{3·9}$$

it follows from theorem A that there exist solutions of (3·2), namely $W_1(\xi)$ and $W_2(\xi)$, such that if $\xi_1$ lies in $\mathbf{Z}''_\xi$ and $\xi_2$ lies in $\mathbf{Z}'_\xi$, then for large positive values of $u$,

$$W_1(\xi) \sim \mathrm{e}^{u\xi_1}\sum_{s=0}^{\infty}\frac{T_s(\xi_1)}{u^s}, \quad W_2(\xi_2) \sim \mathrm{e}^{-u\xi_2}\sum_{s=0}^{\infty}\frac{T_s(\xi_2)}{(-u)^s}, \tag{3·10}$$

and these expansions hold uniformly with respect to $\xi$.

The matching of the Legendre functions with $W_1(\xi_1)$ and $W_2(\xi_2)$ is completed in the usual way, by the examination of the functions at $z = 1$ ($\xi = -\infty$) and $|z| = \infty$ ($\xi = +\infty$), from (2·7) and (2·8). We deduce from (3·10) that for large values of $u$

$$(z^2-\beta^2)^{\frac{1}{4}}\, P_n^{-m}(z_1) \sim \lambda_u \sqrt{\alpha}\, \mathrm{e}^{u\xi_1}\sum_{s=0}^{\infty}\frac{T_s(\xi_1)}{u^s}, \tag{3·11}$$

$$2\,\mathrm{e}^{m\pi\mathrm{i}}\,(z^2-\beta^2)^{\frac{1}{4}}\, Q_n^{-m}(z_2) \sim \mu_u \sqrt{\alpha}\, \mathrm{e}^{-u\xi_2}\sum_{s=0}^{\infty}\frac{T_s(\xi_2)}{(-u)^s}, \tag{3·12}$$

where $\xi_1$ lies in $\mathbf{Z}''_\xi$ and $\xi_2$ lies in $\mathbf{Z}'_\xi$, $\lambda_u$ and $\mu_u$ are functions of $u$ only and are yet to be determined; this is achieved below by first fixing the integration constants in (3·9). The expansions (3·11) and (3·12) are uniformly valid with respect to $z$ for $z_1$ lying in $\mathbf{Z}''_z$ and $z_2$ lying in $\mathbf{Z}'_z$; we have $\mathbf{Z}'_z = \mathbf{Z}_z - \mathbf{\Delta}_z$ and $\mathbf{Z}''_z = \mathbf{Z}'_z - \mathbf{R}_z$, where $\mathbf{R}_z$ consists of the two strips $|\mathscr{I}z| < \delta_1$, $0 < \mathscr{R}z < \beta - \delta_1$ where $\delta_1 \equiv \delta_1(\delta) > 0$.

We now determine $\lambda_u$, $\mu_u$. Let $z \to 1$, $\xi \to -\infty$ in (3·11) and (3·12). It follows from (2·7), (2·10) and (3·8) that

$$\frac{1}{\Gamma(m+1)}(\tfrac{1}{2}p)^m \sim \lambda_u \sum_{s=0}^{\infty}\frac{T_s(-\infty)}{u^s} \sim \lambda_u \sum_{s=0}^{\infty}\frac{\gamma_s}{u^s}, \quad \text{say}, \tag{3·13}$$

$$\Gamma(m)\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}\left(\frac{2}{p}\right)^m \sim \mu_u \sum_{s=0}^{\infty}\frac{T_s(-\infty)}{(-u)^s} \sim \mu_u \sum_{s=0}^{\infty}\frac{\gamma_s}{(-u)^s}. \tag{3·14}$$

Now $T_0(\xi) = 1$ and thus $\quad \ln\left\{\sum_{s=0}^{\infty}\gamma_s(\pm u)^{-s}\right\} \sim \sum_{s=1}^{\infty} c_s(\pm u)^{-s}$, say.

Taking logarithms of (3·13) and (3·14) and adding and subtracting the two expansions we derive

$$\ln \lambda_u \mu_u + 2\sum_{s=1}^{\infty} \frac{c_{2s}}{u^{2s}} \sim \ln \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} - \ln m, \tag{3·15}$$

$$\ln \frac{\mu_u}{\lambda_u} - 2\sum_{s=0}^{\infty} \frac{c_{2s+1}}{u^{2s+1}} \sim \ln 2\pi + T_u + 2R_m, \tag{3·16}$$

where $R_m$ and $T_u$ are given in (2·16) and (2·17). Since the constants $\gamma_s$ are at our disposal, we can set $c_{2r} = 0$ for $r > 1$. All the conditions are now satisfied if we put

$$\lambda_u = \frac{1}{2\pi}\mu_u = \frac{1}{\surd(2\pi m)}\left\{\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}\right\}^{\frac{1}{2}} = \frac{1}{\surd\alpha}\Lambda_u, \quad \text{say.} \tag{3·17}$$

The desired expansions are therefore, from (3·11) and (3·12),

$$P_n^{-m}(z_1) \sim \Lambda_u (z^2-\beta^2)^{-\frac{1}{4}} e^{u\xi_1} \sum_{s=0}^{\infty} \frac{T_s(\xi_1)}{u^s}, \tag{3·18}$$

$$Q_n^{-m}(z_2) \sim \pi\, e^{-m\pi i} \Lambda_u (z^2-\beta^2)^{-\frac{1}{4}} e^{-u\xi_2} \sum_{s=0}^{\infty} \frac{T_s(\xi_2)}{(-u)^s}, \tag{3·19}$$

where $z_1$ lies in $\mathbf{Z}''_z = \mathbf{Z}_z - \mathbf{\Delta}_z - \mathbf{R}_z$ and $z_2$ lies in $\mathbf{Z}'_z = \mathbf{Z}_z - \mathbf{\Delta}_z$, where $\mathbf{\Delta}_z$ is the region $|\mathscr{R}z - \beta| < \delta_1$, $|\mathscr{I}z| < \delta_1$ and $\mathbf{R}_z$ is $|\mathscr{I}z| < \delta_1$, $0 \leqslant \mathscr{R}z < \beta - \delta_1$, $\delta_1 \equiv \delta_1(\delta) > 0$.

*The functions $T_s(\xi)$*

These functions are polynomials of degree $3s$ in $v$, where $v = \alpha z(z^2-\beta^2)^{-\frac{1}{2}}$. This follows since

$$\frac{dv}{d\xi} = \frac{1}{\alpha\beta^2}(v^2-1)(v^2-\alpha^2), \quad f(\xi) = \frac{1}{4\alpha^2\beta^4}(1-v^2)(v^2-\alpha^2)(5v^2-1-\alpha^2), \tag{3·20}$$

and

$$T_{s+1}(v) = -\frac{(v^2-1)(v^2-\alpha^2)}{2\alpha\beta^2}\frac{dT_s}{dv} - \frac{1}{8\alpha\beta^2}\int_1^v (5v^2-1-\alpha^2)\, T_s(v)\, dv + \gamma_{s+1}. \tag{3·21}$$

Thus if we write $dT/d\xi = T'$, $dT/dv = \dot{T}$, we conclude that

$$T_0(v) = 1, \quad 2T_1'(\xi) = f(\xi), \quad T_1(\xi) = -\frac{v}{24\alpha\beta^2}\{5v^2 - 3(1+\alpha^2)\},$$

$$T_2 = -\tfrac{1}{2}T_1' + \int_{-\infty}^{\xi} T_1' T_1\, d\xi + \gamma_2$$

$$= \frac{1}{1152\alpha^2\beta^4}\{385v^6 - 462v^4(1+\alpha^2) + v^2(81\alpha^4 + 522\alpha^2 + 81) - 72\alpha^2(1+\alpha^2)\},$$

$$T_3 = -\tfrac{1}{4}fT_1 - \tfrac{1}{8}f' + [\tfrac{1}{6}T_1^3]_1^v - \frac{1}{8}\int_1^v f^2 \dot{\xi}\, dv + \gamma_3.$$

We can prove that $T_{2p}(v)$ is an even function of $v$ and $T_{2p+1}(v)$ is an odd function of $v$. To show this, it is sufficient to show that $t_{2p+1} = 0$, where $t_s = [T_s(v)]_{v=0}$. Let $z \to 0 + i0$ in (3·19) and apply the formulae (2·11) and (2·18*b*). Thus

$$F(u,\alpha) \equiv \surd(\tfrac{1}{2}u\beta)\left\{\frac{\Gamma(\frac{1}{2}n+\frac{1}{2}m+\frac{1}{2})\,\Gamma(\frac{1}{2}n-\frac{1}{2}m+\frac{1}{2})}{\Gamma(\frac{1}{2}n+\frac{1}{2}m+1)\,\Gamma(\frac{1}{2}n-\frac{1}{2}m+1)}\right\}^{\frac{1}{2}} \sim \sum_{s=0}^{\infty} \frac{t_s}{(-u)^s}. \tag{3·22}$$

From (2·18*a*) we can show that

$$G(u,\alpha) \equiv \ln F(u,\alpha) - \ln F(-u,\alpha) = \ln\left\{1 + \frac{2\cos u\pi}{\sin u\pi - \cos \alpha u\pi}\right\}.$$

Now $G(u,\alpha)$ has an asymptotic expansion in ascending powers of $u^{-1}$ valid for $|\arg u| < \pi$. If $u$ is now chosen so that $\mathscr{I}u = -a$, say, where $a$ is large and positive, it follows that $G(u,\alpha) = 1 + O(\mathrm{e}^{-a(1-\alpha)})$. Thus the series on the right-hand side of (3·22) contains only even powers of $u^{-1}$ and thus $t_{2p+1} = 0$. Hence

$$T_s(-v) = (-1)^s T_s(v). \tag{3·23}$$

When $z = x$ and $\beta < x < 1$, the points $\xi_{1\pm}$ correspond to $x \pm \mathrm{i}0$, where

$$\xi_{1\pm} = \chi_1 \pm \tfrac{1}{2}\mathrm{i}\alpha\pi, \quad \chi_1 = \cosh^{-1}\frac{x}{\beta} - \alpha\cosh^{-1}\left\{\frac{\alpha x}{\beta\surd(1-x^2)}\right\}, \tag{3·24}$$

and $T_s(\xi_{1+}) = T_s(\xi_{1-}) = T_s(\chi_1)$, $v_1 = \alpha x(x^2-\beta^2)^{-\frac{1}{2}}$. When $z = x$ and $0 \leqslant x < \beta$, $\xi_{2\pm}$ and $v_{2\pm}$ correspond to $x \pm \mathrm{i}0$, where

$$\xi_{2\pm} = \pm\mathrm{i}(\chi_2 + \tfrac{1}{2}\alpha\pi), \quad \chi_2 = \cos^{-1}\frac{x}{\beta} - \alpha\cos^{-1}\left\{\frac{\alpha x}{\beta\surd(1-x^2)}\right\}, \tag{3·25}$$

$$v_{2\pm} = \mp\frac{\mathrm{i}\alpha x}{\surd(\beta^2-x^2)} = \mp\mathrm{i}v_2, \quad \text{say}. \tag{3·26}$$

Let
$$\left.\begin{aligned} \mathscr{T}_{2p}(v_2) &= T_{2p}(\xi_{2-}) = T_{2p}(\xi_{2+}), \\ \mathscr{T}_{2p+1}(v_2) &= (-\mathrm{i})\,T_{2p+1}(\xi_{2-}) = \mathrm{i}T_{2p+1}(\xi_{2+}). \end{aligned}\right\} \tag{3·27}$$
and

The functions $T_s(\xi)$ have singularities at $z = \beta$ and $\mathscr{T}_s(v_2)$ are real valued functions.

*Expansions for* $\mathrm{P}_n^{-m}(x)$, $\mathrm{Q}_n^{-m}(x)$

These expansions differ according as (i) $\beta + \delta_1 \leqslant x \leqslant 1$ or (ii) $0 \leqslant x \leqslant \beta - \delta_1$. In case (i) we apply (2·2) and (2·3) to (3·18) and (3·19) immediately to give

$$\mathrm{P}_n^{-m}(x) \sim \Lambda_u (x^2-\beta^2)^{-\frac{1}{4}}\,\mathrm{e}^{u\chi_1}\sum_{s=0}^{\infty}\frac{T_s(v_1)}{u^s}, \tag{3·28}$$

$$\mathrm{Q}_n^{-m}(x) \sim \pi\Lambda_u (x^2-\beta^2)^{-\frac{1}{4}}\,\mathrm{e}^{-u\chi_1}\sum_{s=0}^{\infty}\frac{T_s(v_1)}{(-u)^s}, \tag{3·29}$$

where $\Lambda_u$ and $\chi_1$ are given in (3·17) and (3·24). The expansion (3·19) is valid for $0 \leqslant x \leqslant \beta - \delta_1$ but (3·18) is not valid for these values of $x$. However, the use of (2·3), (2·4), (3·19) and (3·23) give

$$\mathrm{P}_n^{-m}(x) \sim 2\Lambda_u(\beta^2-x^2)^{-\frac{1}{4}}\left\{\cos(u\chi_2 - \tfrac{1}{4}\pi)\sum_{s=0}^{\infty}\frac{\mathscr{T}_{2s}(v_2)}{u^{2s}} + \sin(u\chi_2 - \tfrac{1}{4}\pi)\sum_{s=0}^{\infty}\frac{\mathscr{T}_{2s+1}(v_2)}{u^{2s+1}}\right\}, \tag{3·30}$$

$$\mathrm{Q}_n^{-m}(x) \sim \pi\Lambda_u(\beta^2-x^2)^{-\frac{1}{4}}\left\{\cos(u\chi_2 + \tfrac{1}{4}\pi)\sum_{s=0}^{\infty}\frac{\mathscr{T}_{2s}(v_2)}{u^{2s}} + \sin(u\chi_2 + \tfrac{1}{4}\pi)\sum_{s=0}^{\infty}\frac{\mathscr{T}_{2s+1}(v_2)}{u^{2s+1}}\right\}. \tag{3·31}$$

*Expansions for derivatives*

Olver's theory shows that we can differentiate the above expansions with respect to $z$ in order to obtain expansions for the derivatives of the Legendre functions. Thus

$$\frac{\mathrm{d}}{\mathrm{d}z}P_n^{-m}(z_1) \sim u\Lambda_u\frac{(z^2-\beta^2)^{\frac{1}{4}}}{z^2-1}\,\mathrm{e}^{u\xi_1}\sum_{s=0}^{\infty}\frac{T_s^{(1)}(\xi_1)}{u^s}, \tag{3·32}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}Q_n^{-m}(z_2) \sim -\pi\,\mathrm{e}^{-m\pi\mathrm{i}}\,u\Lambda_u\frac{(z^2-\beta^2)^{\frac{1}{4}}}{z^2-1}\,\mathrm{e}^{-u\xi_2}\sum_{s=0}^{\infty}\frac{T_s^{(1)}(\xi_2)}{(-u)^s}, \tag{3·33}$$

where $$T_s^{(1)} = T_s - \frac{1-v^2}{\alpha\beta^2}\left\{\tfrac{1}{2}vT_{s-1} + (v^2-\alpha^2)\frac{dT_{s-1}}{dv}\right\}, \tag{3·34}$$

and for $z = x$, where $\beta+\delta_1 \leqslant x \leqslant 1$, we have

$$\frac{d}{dx}P_n^{-m}(x) \sim -u\Lambda_u \frac{(x^2-\beta^2)^{\frac{1}{2}}}{1-x^2} e^{u\chi_1} \sum_{s=0}^{\infty} \frac{T_s^{(1)}(\chi_1)}{u^s}, \tag{3·35}$$

$$\frac{d}{dx}Q_n^{-m}(x) \sim \pi u\Lambda_u \frac{(x^2-\beta^2)^{\frac{1}{2}}}{1-x^2} e^{-u\chi_1} \sum_{s=0}^{\infty} \frac{T_s^{(1)}(\chi_1)}{(-u)^s}, \tag{3·36}$$

and when $0 \leqslant x \leqslant \beta-\delta_1$ we have

$$\frac{d}{dx}P_n^{-m}(x) \sim 2u\Lambda_u \frac{(\beta^2-x^2)^{\frac{1}{2}}}{(1-x^2)}\left\{\cos(u\chi_2-\tfrac{3}{4}\pi)\sum_{s=0}^{\infty}\frac{\mathscr{T}_{2s}^{(1)}(v_2)}{u^{2s}} + \sin(u\chi_2-\tfrac{3}{4}\pi)\sum_{s=0}^{\infty}\frac{\mathscr{T}_{2s+1}^{(1)}(v_2)}{u^{2s+1}}\right\}, \tag{3·37}$$

$$\frac{d}{dx}Q_n^{-m}(x) \sim 2\pi u\Lambda_u \frac{(\beta^2-x^2)^{\frac{1}{2}}}{1-x^2}\left\{\cos(u\chi_2-\tfrac{1}{4}\pi)\sum_{s=0}^{\infty}\frac{\mathscr{T}_{2s}^{(1)}(v_2)}{u^{2s}} + \sin(u\chi_2-\tfrac{1}{4}\pi)\sum_{s=0}^{\infty}\frac{\mathscr{T}_{2s+1}^{(1)}(v_2)}{u^{2s+1}}\right\}, \tag{3·38}$$

where $$\mathscr{T}_s^{(1)} = \mathscr{T}_s - (-1)^s\frac{1+v_2^2}{\alpha\beta^2}\left\{\tfrac{1}{2}v_2\mathscr{T}_{s-1} + (v_2^2+\alpha^2)\frac{d\mathscr{T}_{s-1}}{dv_2}\right\}.$$

*Previous results.* No asymptotic expansions have been obtained before for the Legendre functions of large degree and order. Single-term approximations for $P_n^{-m}(x)$ have been given in Jeffreys & Jeffreys (1950, p. 658). By applying Stirling's formula to $\Lambda_u$ it is possible to show that the leading terms of (3·28) and (3·30) are identical with these approximations.

*Conclusion.* The expansions (3·18) for $P_n^{-m}(z)$ and (3·32) for $d\{P_n^{-m}(z)\}/dz$ are valid uniformly with respect to $z$ everywhere in the cut half-plane $\mathscr{R}z \geqslant 0$ except for the domain $\mathbf{\Delta}_z$ surrounding $z=\beta$ and the strip $|\mathscr{I}z| < \delta_1$, $0 \leqslant \mathscr{R}z \leqslant \beta-\delta_1$. The expansions (3·19) and (3·23) for $Q_n^{-m}(z)$ and its derivative are valid, however, in the whole cut half-plane except for the domain $\mathbf{\Delta}_z$. Expansions on the cut $0 \leqslant x < \beta-\delta_1$, $\beta+\delta_1 \leqslant x \leqslant 1$ have also been derived. For other ranges of $\arg z$, the continuation formulae (2·14) and (2·15) can be used.

## 4. Expansions for the Legendre functions in terms of Airy functions

The expansions developed in this section are obtained from an examination of the equation (1·3), as was the case with the exponential-type expansions of § 3. The coefficient of $u^2w$ in (1·3) has double poles at $z=\pm 1$, the point at infinity is a regular singularity of (1·3) and the points $z=\pm\beta$ are turning points of (1·3). At these turning points the value of $dz/d\xi$ in (3·1) becomes infinite. Since the $z$ plane is cut from $+1$ to $-\infty$, the equation (1·3) is a particular example, as was mentioned in § 1, of a differential equation considered in an earlier paper (I, § 5). To obtain expansions valid at the turning point $z=\beta+i0$ we make the $z$-$\zeta$ transformation (see I, (2·5))

$$\left.\begin{aligned} \frac{dz}{d\zeta} &= -\frac{z^2-1}{\surd(z^2-\beta^2)}\zeta^{\frac{1}{2}}, \quad W = \left(\frac{dz}{d\zeta}\right)^{-\frac{1}{2}} w, \\ \rho &= -\xi + \tfrac{1}{2}i\alpha\pi = -\int_\beta^z \frac{\surd(z^2-\beta^2)}{z^2-1}dz, \quad \zeta = (\tfrac{3}{2}\rho)^{\frac{2}{3}}, \end{aligned}\right\} \tag{4·1}$$

where $\rho$ and $\xi$ are the same as in (3·4). The lower limit of the integral is $\beta+i0$. Then $W(\zeta)$ satisfies the equation

$$\frac{d^2W}{d\zeta^2} = \{u^2\zeta + f_1(\zeta)\}W, \tag{4·2}$$

where, if $f(\xi)$ is the function given in (3·7),

$$f_1(\zeta) = \frac{5}{16\zeta^2} + \zeta f(\xi)$$
$$= \frac{5}{16\zeta^2} + \frac{\zeta(z^2-1)}{4(z^2-\beta^2)^3}\{z^2(4\alpha^2-1)+(1-\alpha^4)\}. \quad (4·3)$$

*The z-ζ transformation* (4·1)

Before asymptotic solutions of (4·3) can be investigated it is first necessary to examine the region in the $\zeta$ plane corresponding to the cut $z$ plane, and then secondly to examine the behaviour of $f_1(\zeta)$ in this region. To examine the $z$-$\zeta$ transformation we set (see (3·4) and (3·5))

$$\lambda = \frac{\alpha}{\beta}(z^{-2}-1)^{-\frac{1}{2}}, \quad \rho = \alpha\cosh^{-1}\lambda - \cosh^{-1}\frac{z}{\beta}. \quad (4·4)$$

Let $\mathbf{I}_{1z}$ denote the quadrant $\mathscr{R}z \geqslant 0$, $\mathscr{I}z \geqslant 0$ in the cut $z$ plane; and in this cut plane let $\mathbf{I}_{2z}, \mathbf{I}_{3z}, \mathbf{I}_{4z}$ denote the three other quadrants (in order) reached from $\mathbf{I}_{1z}$ by a counterclockwise rotation about the origin. Let $\mathbf{I}_z$ denote the cut $z$ plane. Then $\mathbf{I}_{1\lambda}$ and $\mathbf{I}_{2\lambda}$ are the quadrants $\mathscr{R}\lambda \geqslant 0$, $\mathscr{I}\lambda \geqslant 0$ and $\mathscr{R}\lambda \leqslant 0$, $\mathscr{I}\lambda \geqslant 0$ respectively, with a cut $E_\lambda F_\lambda H_\lambda A_\lambda$ from $\alpha\beta^{-1}e^{\frac{1}{2}\pi i}$ to $\infty\, e^{\frac{1}{2}\pi i}$. $\mathbf{I}_{3\lambda}$ and $\mathbf{I}_{4\lambda}$ are the same quadrants as $\mathbf{I}_{1\lambda}$ and $\mathbf{I}_{2\lambda}$ and are taken as lying on a sheet of a Riemann surface reached from $\mathbf{I}_{1\lambda}$ by crossing $H_\lambda A_\lambda$ from the right as indicated by the broken lines in figure 4. $\mathbf{I}_{1\rho}$ is the strip $\mathscr{R}\rho < 0$, $-\frac{1}{2}\pi(1-\alpha) \leqslant \mathscr{I}\rho \leqslant \frac{1}{2}\alpha\pi$ together with the strip $\mathscr{R}\rho \geqslant 0$, $0 \leqslant \mathscr{I}\rho \leqslant \frac{1}{2}\alpha\pi$. $\mathbf{I}_{2\rho}$ is the reflexion of $\mathbf{I}_{1\rho}$ in the line $\mathscr{I}\rho = -\frac{1}{2}(1-\alpha)\,\pi$ and $\mathbf{I}_{4\rho}$, $\mathbf{I}_{3\rho}$ are the reflexions of $\mathbf{I}_{1\rho}$, $\mathbf{I}_{2\rho}$, in that order, in the line $\mathscr{I}\rho = \frac{1}{2}\alpha\pi$. Finally, $\mathbf{I}_z$ is mapped conformally into a domain in the $\zeta$ plane consisting of a region bounded by the curves (figure 6) $A_\zeta Q_\zeta$, $Q_\zeta F_\zeta$, $F_{1\zeta}Q_{1\zeta}$, $Q_{1\zeta}B_{1\zeta}$, $B_{1\zeta}A_\zeta$, together with the strips $\mathbf{I}'_{2\zeta}$ and $\mathbf{L}'_{3\zeta}$, say, bounded respectively by $E_\zeta Q_\zeta D_\zeta E_\zeta$ and $E_{1\zeta}Q_{1\zeta}D_{1\zeta}E_{1\zeta}$. The curves $F_\zeta Q_\zeta$ and $F_{1\zeta}Q_{1\zeta}$ are represented by

$$\zeta = \{\tfrac{3}{2}(ct + \mathrm{i}d\pi)\}^{\frac{2}{3}} \quad (0 \leqslant t < \infty), \quad (4·5)$$

where $c = -1$ and $d = -(1-\frac{1}{2}\alpha)$, $+(1+\frac{1}{2}\alpha)$ respectively. For the curves $Q_\zeta E_\zeta$, $E_\zeta D_\zeta$, $A_\zeta B_{1\zeta}$, $D_{1\zeta}E_{1\zeta}$, $E_{1\zeta}Q_{1\zeta}$ we have $c = +1$ in (4·5) and for $d$ we have the values $-(1-\frac{1}{2}\alpha)$, $-(1-\alpha)$, $\alpha$, $+1$, $(1+\frac{1}{2}\alpha)$ respectively.

We denote by $\mathbf{I}'$, $\mathbf{I}'_2$, $\mathbf{I}'_3$ the domains obtained by deleting from $\mathbf{I}$, $\mathbf{I}_2$, $\mathbf{I}_3$ the regions corresponding to $\mathbf{L}'_{2\zeta}$, $\mathbf{L}'_{3\zeta}$. Then $\mathbf{L}'_{2z}$ and $\mathbf{L}'_{3z}$ are the pear-shaped domains, bounded by the $z$ axis and the lines surrounding $E_z$ and $E_{1z}$ shown in figure 3. Let $\mathbf{L}'_z = \mathbf{L}'_{2z} + \mathbf{L}'_{3z}$ and let $\mathbf{L}_z$ denote the reflexion of $\mathbf{L}'_z$ in the imaginary $z$ axis. The curve $D_z Q_z$, being part of the boundary of $\mathbf{L}'_{2z}$, corresponds to a curve $D_\sigma Q_\sigma$ in the $\sigma$-plane where $\sigma = \cosh^{-1}(z/\beta)$. $D_\sigma Q_\sigma$ is given by

$$\sigma = t + \mathrm{i}\pi - \tfrac{1}{2}\mathrm{i}\cos^{-1}[\tfrac{1}{2}\operatorname{cosech}(2\alpha^{-1}t)\,\{\tau\sinh 2(\alpha^{-1}+1)\,t + \tau^{-1}\sinh 2(\alpha^{-1}-1)\,t\}], \quad (4·6)$$

where $t = \mathscr{R}\sigma$, $\tau = (1-\alpha)/(1+\alpha)$ and $0 \leqslant t \leqslant t_1$ where $t_1$ is given by $\tanh t_1 = \alpha\coth(\alpha^{-1}t_1)$. Then $D_z Q_z$ is given by

$$2z = -\sinh 2t[\{\coth^2 t - 2\alpha\coth t\,\coth(2\alpha^{-1}t) + \alpha^2\}^{\frac{1}{2}}$$
$$-\mathrm{i}\{2\alpha\tanh t\,\coth(2\alpha^{-1}t) - \alpha^2 - \tanh^2 t\}^{\frac{1}{2}}], \quad (4·7)$$

where $0 \leqslant t \leqslant t_1$; $Q_z$ has affix $\beta \cosh t_1 \, e^{\pi i}$ and lies to the left of $E_z$. The boundary of $\mathbf{L}'_z$ leaves $D_z$ at an angle of $\frac{2}{3}\pi$ with the real axis, and intersects the real axis at $Q_z$ and $Q_{1z}$ at right angles. The domain $\mathbf{L}_z$ has similar properties, and $P_z$ has affix $\beta \cosh t_1$; $\mathbf{L}_z$ is a particular example of the domain $\mathbf{T}_z$ of I, § 3.

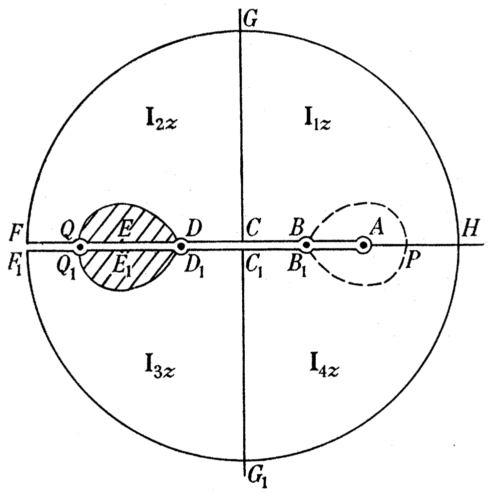


FIGURE 3. $z$ plane.

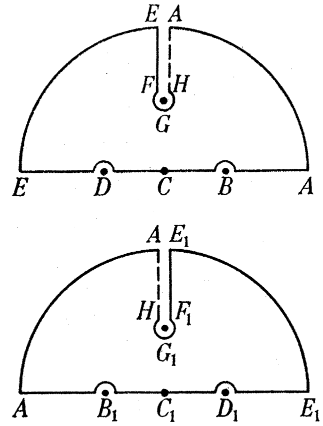


FIGURE 4. $\lambda$ plane.

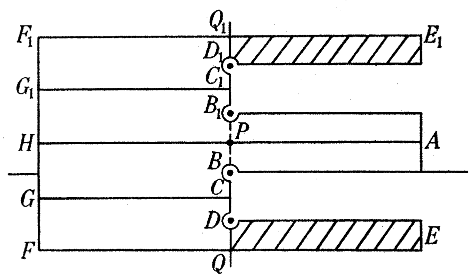


FIGURE 5. $\rho$ plane.

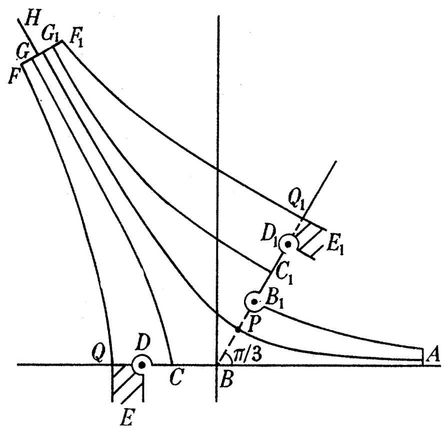


FIGURE 6. $\zeta$ plane.

*Further consideration of the $\zeta$ plane*

We obtain the asymptotic expansions of the solutions of (4·2) by the application of a certain theorem, theorem B, proved by Olver (1954*b*, § 5). To apply this theorem it is necessary that $f_1(\zeta)$ in (4·2) should be a regular function of $\zeta$ in an open simply-connected domain $\mathbf{D}$; if $\mathbf{D}$ extends to infinity we require that $f_1(\zeta) = O(|\zeta|^{-\frac{1}{2}-\kappa})$ $(\kappa > 0)$, as $|\zeta| \to \infty$ in $\mathbf{D}$ (Olver 1956, appendix). We also require that the distance between the boundary lines of $\mathbf{D}$ should not shrink to zero as $|\zeta| \to \infty$ in $\mathbf{D}$. We are therefore not able to apply theorem B to (4·2) in the domain $\mathbf{I}_\zeta$ as it stands, since the distance between the boundary curves of $\mathbf{I}_\zeta$ tends to zero as $|\zeta| \to \infty$ in $\mathbf{I}_\zeta$. To obviate this difficulty a Riemann surface on the $z$ plane is constructed to correspond to the $\zeta$ plane cut in a certain way described below.

The sector $0 < \arg \zeta < \frac{1}{3}\pi$ corresponds to $\mathbf{L}_z$ and the pear-shaped domains which we denote by $\mathbf{L}_z^{-j}$, $j = 1, 2, \ldots$, and which are considered as lying below $\mathbf{L}_z$ and reached by successive clockwise rotations about $z = 1$ across $A_{1z}B_{1z}$; for completeness we write $\mathbf{L}_z = \mathbf{L}_z^0$.

To the sector $\frac{1}{3}\pi < \arg\zeta < \frac{2}{3}\pi$ corresponds a $z$ surface consisting of an infinite number of sheets $\mathbf{I}_z^{-j}$, $j = 1, 2, \ldots$, say, lying below $\mathbf{I}_z$, reached by successive clockwise rotations about $z = 0$ across $Q_{1z}F_{1z}$. The domains corresponding to $\mathbf{L}_z$ and $\mathbf{L}'_z$ are deleted from $\mathbf{I}_z^{-j}$. Let $\mathbf{I}_z^0 = \mathbf{I}'_z - \mathbf{L}_z = \mathbf{I}_z - \mathbf{L}_z - \mathbf{L}'_z$. Then the $\mathbf{I}_z^{-j}$ are linked across the cut $Q_{1z}F_{1z}$ so that if $A_z^{-j}$, $B_z^{-j} \ldots$ denote the points $A_z$, $B_z$, ... in $\mathbf{I}_z^{-j}$, then $F_{1z}Q_{1z} \equiv F_{1z}^0 Q_{1z}^0$ in $\mathbf{I}_z^0$ is joined to $F_z^{-1}Q_z^{-1}$ in $\mathbf{I}_z^{-1}$ and so on; $\mathbf{I}_z^{-j}$ is not connected with $\mathbf{L}_z^{-j}$.

The sector $\frac{2}{3}\pi < \arg\zeta < \pi$ corresponds to the $z$ surface consisting of the sheets $\mathbf{I}_z^j$, $j = 1, 2, \ldots$, which lie above $\mathbf{I}_z$, and are reached by counterclockwise rotations about $z = 0$ across the cut $Q_zF_z$. The domains corresponding to $\mathbf{L}_z$ and $\mathbf{L}'_z$ are deleted from $\mathbf{I}_z^j$.

Finally, $\mathscr{I}\zeta < 0$ corresponds to a combination of sheets, $\mathbf{J}_z^0$, $\mathbf{J}_z^{\pm j}$, $\mathbf{L}_z^j$, $j = 1, 2, \ldots$; $\mathbf{J}_z$ and $\mathbf{J}'_z$ are cut in exactly the same way as $\mathbf{I}_z$ and $\mathbf{I}'_z$; they lie above $\mathbf{I}_z$ and are joined to $\mathbf{I}_z$ along $D_zC_zB_zA_z$. The sheets $\mathbf{J}_z^{\pm j}$ $(j = 1, 2, \ldots)$ are related to $\mathbf{J}'_z$ in exactly the same way as $\mathbf{I}_z^{\pm j}$ are related to $\mathbf{I}'_z$. $\mathbf{L}_z^j$ $(j = 1, 2, \ldots)$ are reached by successive counterclockwise rotations about $z = 1$, and since $\mathbf{L}_z^1$ lies in $\mathbf{J}'_z$ we have $\mathbf{J}_z^0 = \mathbf{J}'_z - \mathbf{L}_z^1$. The domains corresponding to $\mathbf{L}_z$ and $\mathbf{L}'_z$ are deleted from $\mathbf{J}_z^{\pm j}$.

Then the $z$ surface,
$$\mathbf{D}_z = \sum_{j=-\infty}^{\infty} (\mathbf{I}_z^j + \mathbf{J}_z^j + \mathbf{L}_z^j),$$
is mapped on to the whole $\zeta$ plane with the three cuts
$$\zeta = t\,e^{\pm\frac{1}{3}\pi i}, \quad (\tfrac{3}{2}\pi\alpha)^{\frac{2}{3}} \leqslant t < \infty, \quad -\infty < \zeta \leqslant -\{\tfrac{3}{2}\pi(1-\tfrac{1}{2}\alpha)\}^{\frac{2}{3}};$$
and $z \equiv z(\zeta)$ is a regular function of $\zeta$ in $\mathbf{D}_\zeta$ and $\mathbf{D}_\zeta$ is an open simply-connected domain in which $\zeta = 0$ is an interior point. We note that to obtain a $z$ surface corresponding to the whole $\zeta$ plane, it is necessary to exclude $\mathbf{L}'_z$ from all the sheets of $\mathbf{D}_z$.

*The behaviour of $f_1(\zeta)$ in $\mathbf{D}_\zeta$*

The function $f_1(\zeta)$ is given by (4·3). From the results of § 3 (see (3·7) and (3·8)) we conclude that $f_1(\zeta) = O(|\zeta|^{-2})$ as $|\zeta| \to \infty$ in $\mathbf{D}_\zeta$. The point $z = \beta$ corresponds to $\zeta = 0$ and as $\zeta \to 0$,
$$\left.\begin{aligned} &(1-\beta^2/z^2)^{\frac{1}{2}} \sim (2\alpha^2/\beta^2)^{\frac{1}{3}}\zeta^{\frac{1}{2}}\{1+d_1\zeta+d_2\zeta^2+O(\zeta^3)\} \\ \text{where}\quad &d_1 = -\frac{1}{10}\left(\frac{2\alpha^2}{\beta^2}\right)^{\frac{5}{3}}\left(\frac{1-\alpha^4}{\alpha^4}\right), \quad d_2 = 4d_1^2 - \frac{1}{14}\left(\frac{2\alpha^2}{\beta^2}\right)^{\frac{7}{3}}\left(\frac{1-\alpha^6}{\alpha^6}\right). \end{aligned}\right\} \tag{4·8}$$
By direct calculation it can then be shown that $f_1(\zeta) = \text{constant} + O(\zeta)$ as $\zeta \to 0$, and hence $f_1(\zeta)$ is regular throughout $\mathbf{D}_\zeta$.

Let us now define a domain $\mathbf{D}'_\zeta = \mathbf{D}_\zeta - \mathbf{X}_{1\zeta} - \mathbf{X}_{2\zeta}$, where $\mathbf{X}_{1\zeta}$, $\mathbf{X}_{2\zeta}$ are the strips
$$\left.\begin{aligned} \mathbf{X}_{1\zeta}&: \quad \mathscr{R}\zeta\,e^{\pm\frac{1}{3}\pi i} \geqslant (\tfrac{3}{2}\pi\alpha)^{\frac{2}{3}} - \delta, \quad |\mathscr{I}\zeta\,e^{\pm\frac{1}{3}\pi i}| < \delta, \\ \mathbf{X}_{2\zeta}&: \quad \mathscr{R}\zeta \leqslant -\{\tfrac{3}{2}\pi(1-\tfrac{1}{2}\alpha)\}^{\frac{2}{3}} + \delta, \quad |\mathscr{I}\zeta| < \delta. \end{aligned}\right\} \tag{4·9}$$
Then $\mathbf{X}_{1\zeta}$ gives a strip on the lower side of the cut $z = x$, $-\beta \leqslant x \leqslant \beta$ and $\mathbf{X}_{1\zeta}$ and $\mathbf{X}_{2\zeta}$ give strips around the boundary of $\mathbf{L}'_z$ together with strips around the boundaries of $\mathbf{I}_z^j$, $\mathbf{L}_z^j$, $\mathbf{J}_z^j$.

*The application of theorem B in $\mathbf{D}'_\zeta$*

It is now necessary to define the domains $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$ of theorem B. If the points $a_1$, $a_2$, $a_3$ of this theorem are taken to be the points $\zeta = +\infty$, $\zeta = \infty\,e^{-\frac{2}{3}\pi i}$, $\zeta = \infty\,e^{\frac{2}{3}\pi i}$ in the $\zeta$ plane, then $\mathbf{D}'_\zeta = \mathbf{D}_1 = \mathbf{D}_2 = \mathbf{D}_3$. Then we conclude from theorem B that there exist solutions of

(4·2) which, for large positive values of $u$, have asymptotic expansions which are in terms of $\mathrm{Ai}\,(u^{\frac{2}{3}}\zeta)$, $\mathrm{Ai}\,(u^{\frac{2}{3}}\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}\,\zeta)$, $\mathrm{Ai}\,(u^{\frac{2}{3}}\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\,\zeta)$. Matching the Legendre functions to the solutions of (4·2) in the usual way, we derive

$$P_n^{-m}(z) \sim C_u\left(\frac{\zeta}{z^2-\beta^2}\right)^{\frac{1}{4}}\left\{\mathrm{Ai}\,(u^{\frac{2}{3}}\zeta)\sum_{s=0}^{\infty}\frac{E_s(\zeta)}{u^{2s}}+\frac{\mathrm{Ai}'\,(u^{\frac{2}{3}}\zeta)}{u^{\frac{4}{3}}}\sum_{s=0}^{\infty}\frac{F_s(\zeta)}{u^{2s}}\right\}, \tag{4·10}$$

$$Q_n^{-m}(z) \sim D_u\left(\frac{\zeta}{z^2-\beta^2}\right)^{\frac{1}{4}}\left\{\mathrm{Ai}\,(u^{\frac{2}{3}}\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\,\zeta)\sum_{s=0}^{\infty}\frac{E_s(\zeta)}{u^{2s}}+\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\frac{\mathrm{Ai}'\,(u^{\frac{2}{3}}\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\,\zeta)}{u^{\frac{4}{3}}}\sum_{s=0}^{\infty}\frac{F_s(\zeta)}{u^{2s}}\right\}, \tag{4·11}$$

where $C_u$ and $D_u$ are functions of $u$ only and $E_0(\zeta) = 1$,

$$\left.\begin{aligned} F_s(\zeta) &= \tfrac{1}{2}\zeta^{-\frac{1}{2}}\int_0^{\zeta} t^{-\frac{1}{2}}\{f_1(t)\,E_s(t)-E_s''(t)\}\,\mathrm{d}t,\\ E_{s+1}(\zeta) &= -\tfrac{1}{2}F_s'(\zeta)+\int_{-\infty}^{\zeta} f_1(t)\,F_s(t)\,\mathrm{d}t+\alpha_{s+1}, \end{aligned}\right\} \tag{4·12}$$

and $\alpha_{s+1}$, $s\geqslant 0$ are integration constants which we suppose are real. The asymptotic expansion in terms of $\mathrm{Ai}\,(u^{\frac{2}{3}}\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}\,\zeta)$ will be relevant for a Legendre function which is bounded at infinity in $\mathbf{J}_z$, but this function is not of interest here.

We now determine $C_u$ and $D_u$. Let $z = x+\mathrm{i}0$, where $\beta\leqslant x\leqslant 1$; then the point $z$ lies on $B_zA_z$, and $\arg(z-1) = \pi$ and $\zeta$, $E_s(\zeta)$, $F_s(\zeta)$, $\mathrm{Ai}\,(u^{\frac{2}{3}}\zeta)$, $\mathrm{e}^{-\frac{1}{2}m\pi\mathrm{i}}P_n^{-m}(z)$ are all real. Hence from (4·10)

$$C_u = \mathrm{e}^{\frac{1}{2}m\pi\mathrm{i}}\,C_u', \tag{4·13}$$

where $C_u'$ is real. For this same value of $x$ let $x\to 1-0$, $\zeta\to+\infty$. Applying (2·7), (2·10), (3·8) and Olver (1954*b*, (4·3)) to the expression (4·11) above, we conclude that $D_u'$ is a real quantity, where

$$D_u' = \frac{1}{\pi}\exp\left(\tfrac{3}{2}m\pi\mathrm{i}+\tfrac{1}{6}\pi\mathrm{i}\right)D_u. \tag{4·14}$$

We now show that $C_u' = D_u'$. From Watson (1944, p. 80) and Olver (1954*b*, (4·2)) we have that

$$\mathrm{Ai}\,(t\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}) = \mathrm{Ai}\,(t)+\frac{1}{\pi\sqrt{3}}\,\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}\sqrt{t}\,I_{\frac{1}{3}}(\tfrac{2}{3}t^{\frac{3}{2}}). \tag{4·15}$$

For $t$ real the functions $\mathrm{Ai}\,(t)$ and $I_{\frac{1}{3}}(\frac{2}{3}t^{\frac{3}{2}})$ in (4·15) are real. Substituting (2·10), (4·14) and (4·15) into (4·11), and multiplying throughout by $\mathrm{e}^{-\frac{1}{6}\pi\mathrm{i}}$ we derive

$$C_1\,\mathrm{e}^{\frac{1}{2}m\pi\mathrm{i}}\{C_2P_n^m(z)-P_n^{-m}(z)\}\sim D_u'\{\mathrm{e}^{-\frac{1}{6}\pi\mathrm{i}}\,S_1+S_2\}, \tag{4·16}$$

where $C_1$ and $C_2$ are real constants and $S_1$ and $S_2$ are real asymptotic expansions. On $B_zA_z$ we have $\mathrm{e}^{\pm\frac{1}{2}m\pi\mathrm{i}}P_n^{\pm m}(x+\mathrm{i}0) = \mathrm{P}_n^{\pm m}(x)$. If the real and imaginary parts of both sides of (4·16) are separated for $z = x+\mathrm{i}0$, an asymptotic expansion for $\mathrm{P}_n^{-m}(x)$ is obtained. This expansion is identical with that derived from (4·10), using (4·13) and (2·2), except that the coefficient $C_u'$ is replaced by $D_u'$. Since such asymptotic expansions are unique, we conclude that $C_u' = D_u'$.

To find $C_u'$ we let $z\to 1-0$ in (4·10) and (4·11). Using (2·7), (2·10) and (3·8) we derive

$$\frac{1}{\Gamma(m+1)}\left(\frac{p}{2}\right)^m \sim C_u''\left\{\sum_{s=0}^{\infty}\left(\frac{\alpha_r}{u^{2r}}-\frac{\beta_r}{u^{2r+1}}\right)\right\}, \tag{4·17}$$

$$\frac{\Gamma(m)\,\Gamma(n-m+1)}{2\pi\Gamma(n+m+1)}\left(\frac{2}{p}\right)^m \sim C_n''\left\{\sum_{s=0}^{\infty}\left(\frac{\alpha_r}{u^{2r}}+\frac{\beta_r}{u^{2r+1}}\right)\right\}, \tag{4·18}$$

where $C''_u = \surd(2\alpha\pi u^{\frac{1}{3}}).\ C'_u$; the constants $\alpha_r$ are given in (4·12) and $\beta_r = \lim_{\zeta\to+\infty} \zeta^{-\frac{1}{4}} F_r(\zeta)$; this limit exists by virtue of lemma 2 of Olver (1954*b*, p. 319). Since $\alpha_0 = 1$, we conclude that

$$\ln\left\{\sum_{s=0}^{\infty}\left(\frac{\alpha_s}{u^{2s}}+\frac{\beta_s}{u^{2s+1}}\right)\right\} \sim \sum_{s=1}^{\infty}\frac{\gamma_s}{u^s},$$

where $\gamma'_r$ involves the constants $\alpha_0, \alpha_1, \ldots, \alpha_s$; $\beta_0, \beta_1, \ldots, \beta_{s-1}$ if $r = 2s$, and $\alpha_0, \alpha_2, \ldots, \alpha_s$; $\beta_0, \beta_1, \ldots, \beta_s$ if $r = 2s+1$. The constants $\alpha_j$ $(j \geqslant 1)$ can be chosen arbitrarily. Logarithms of (4·17) and (4·18) are now taken and the resulting expressions added. We then set $\gamma'_{2s} = 0$ (compare (3·17)). Under these circumstances $\alpha_s$ is specified in terms of $\alpha_j, \beta_j$ $(j = 0, 1, 2, \ldots, s-1)$, and $C''_u = \Lambda_u$ defined in (3·17). Hence

$$\mathrm{e}^{-\frac{1}{2}m\pi\mathrm{i}}\, P_n^{-m}(z) \sim \left\{\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}\right\}^{\frac{1}{2}} \left(\frac{4\zeta}{z^2-\beta^2}\right)^{\frac{1}{4}} \left\{\frac{\mathrm{Ai}\,(u^{\frac{2}{3}}\zeta)}{u^{\frac{1}{3}}} \sum_{s=0}^{\infty}\frac{E_s(\zeta)}{u^{2s}} + \frac{\mathrm{Ai}'\,(u^{\frac{2}{3}}\zeta)}{u^{\frac{5}{3}}} \sum_{s=0}^{\infty}\frac{F_s(\zeta)}{u^{2s}}\right\}, \quad (4·19)$$

$$\mathrm{e}^{\frac{3}{2}\pi m\mathrm{i}+\frac{1}{6}\pi\mathrm{i}}\, Q_n^{-m}(z) \sim \pi\left\{\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}\right\}^{\frac{1}{2}} \left(\frac{4\zeta}{z^2-\beta^2}\right)^{\frac{1}{4}} \left\{\frac{\mathrm{Ai}\,(u^{\frac{2}{3}}\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\,\zeta)}{u^{\frac{1}{3}}} \sum_{s=0}^{\infty}\frac{E_s(\zeta)}{u^{2s}} + \mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\frac{\mathrm{Ai}'\,(u^{\frac{2}{3}}\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\,\zeta)}{u^{\frac{5}{3}}} \sum_{s=0}^{\infty}\frac{F_s(\zeta)}{u^{2s}}\right\}. \quad (4·20)$$

The expansions (4·19) and (4·20) are the desired result. The functions $E_s(\zeta)$ and $F_s(\zeta)$ are given by the formulae (4·12) in which the integration constants $\alpha_s$ are specified by the relation

$$\sum_{s=0}^{\infty}\left(\frac{\alpha_s}{u^{2s}}+\frac{\beta_s}{u^{2s+1}}\right) \sim \frac{\surd m}{\surd(2\pi)}\left(\frac{2}{p}\right)^m \Gamma(m)\left\{\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}\right\}^{\frac{1}{2}}. \quad (4·21)$$

*Properties of the asymptotic expansions.* The expansions (4·19) and (4·20) are uniformly valid with respect to $z$ in $\mathbf{D}'_z$. Thus they are valid throughout the $z$ plane cut from $+1$ to $-\infty$ except for (i) the domain $\mathbf{L}'_z$ surrounding the point $z = -1$, and (ii) the strip $-\delta_1 < \mathscr{I}z \leqslant 0$, $|\mathscr{R}z| \leqslant \beta+\delta_1$, $\delta_1 \equiv \delta_1(\delta) > 0$. In both these regions asymptotic expansions can be obtained by use of the continuation formulae (2·12) and (2·13). If $z$ lies in $\mathbf{L}_z$ the expansions hold for *any* value of $\arg(z-1)$. If $z$ lies in the cut plane, but does not lie in the strip (ii) mentioned above or in $\mathbf{L}_z$ or $\mathbf{L}'_z$, the expansions hold for any value of $\arg z$.

The significance of the domains $\mathbf{L}_z$ and $\mathbf{L}'_z$ can now be seen. Inside $\mathbf{L}_z$ the value of $|P_n^{-m}(z)|$ becomes everywhere exponentially small as $u\to\infty$ for all values of $\arg(z-1)$. Outside $\mathbf{L}_z$ the value of $|P_n^{-m}(z)|$ becomes everywhere exponentially large as $u\to\infty$ except for $z = x$ and $|x| < \beta$, when $P_n^{-m}(x)$ is bounded and oscillatory, the zeros of this Legendre function for $-\beta \leqslant x \leqslant \beta$ corresponding to the zeros of $\mathrm{Ai}\,(u^{\frac{2}{3}}\zeta)$ for $\zeta$ lying in the interval $D_\zeta B_\zeta$ in figure 6 (see (4·22) below). As $u\to\infty$ the value of $|Q_n^{-m}(z)|$ becomes exponentially large or small according as $z$ lies inside or outside $\mathbf{L}_z$ and $\mathbf{L}'_z$ except when $z = x$, $|x| < \beta$, when $Q_n^{-m}(x)$ oscillates boundedly (see (4·23) below). Let $z^* = z\,\mathrm{e}^{r\pi\mathrm{i}}$, where $r$ is an integer and $|\arg z| < \pi$, and $z$ does not lie in either $\mathbf{L}_z$ or in $\mathbf{L}'_z$; we suppose that $z^*$ is reached from $z$ by crossing the cut $F_z Q_z$ or $F_{1z} Q_{1z}$. Then the values of $|P_n^{-m}(z^*)|$ and $|Q_n^{-m}(z^*)|$ become exponentially large and small respectively, as $u\to\infty$. This behaviour is borne out by the continuation formulae (2·14) and (2·15). Both Legendre functions are exponentially large as $u\to\infty$ for $z$ in $\mathbf{L}'_z$. The function $P_n^{-m}(-z)$ is the only Legendre function which is exponentially small as $u\to\infty$ for $z$ in $\mathbf{L}'_z$.

*Expansions for the Ferrers functions*

An expansion for $\mathrm{P}_n^{-m}(x)$ is immediate from (4·19) and (2·2). For $\mathrm{Q}_n^{-m}(x)$ we use (2·3) and (4·20); we note from Olver (1954*c*, p. 365) that

$$\mathrm{Bi}(\zeta) = \mathrm{i}\,\mathrm{Ai}(\zeta) + 2\,\mathrm{e}^{-\frac{1}{6}\pi\mathrm{i}}\,\mathrm{Ai}(\zeta\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}).$$

We conclude that

$$\mathrm{P}_n^{-m}(x) \sim \left\{\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}\right\}^{\frac{1}{2}} \left(\frac{4\zeta}{x^2-\beta^2}\right)^{\frac{1}{4}} \left\{\frac{\mathrm{Ai}(u^{\frac{2}{3}}\zeta)}{u^{\frac{1}{3}}} \sum_{s=0}^{\infty} \frac{E_s(\zeta)}{u^{2s}} + \frac{\mathrm{Ai}'(u^{\frac{2}{3}}\zeta)}{u^{\frac{5}{3}}} \sum_{s=0}^{\infty} \frac{F_s(\zeta)}{u^{2s}}\right\}, \tag{4·22}$$

$$\frac{2}{\pi}\mathrm{Q}_n^{-m}(x) \sim \left\{\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}\right\}^{\frac{1}{2}} \left(\frac{4\zeta}{x^2-\beta^2}\right)^{\frac{1}{4}} \left\{\frac{\mathrm{Bi}(u^{\frac{2}{3}}\zeta)}{u^{\frac{1}{3}}} \sum_{s=0}^{\infty} \frac{E_s(\zeta)}{u^{2s}} + \frac{\mathrm{Bi}'(u^{\frac{2}{3}}\zeta)}{u^{\frac{5}{3}}} \sum_{s=0}^{\infty} \frac{F_s(\zeta)}{u^{2s}}\right\}. \tag{4·23}$$

The expansions (3·28) to (3·31) can be derived from (4·22) and (4·23) and the asymptotic expansions for the Airy functions (Olver 1954*c*, appendix (A 6) and (A 8)).

It is not difficult to deduce the form of the Airy-type asymptotic expansions of the Ferrers functions $\mathrm{P}_n^{-m}(z)$, $\mathrm{Q}_n^{-m}(z)$. With these functions the $z$ plane is cut along the real axis from $+1$ to $+\infty$ and from $-1$ to $-\infty$. The same $z$-$\zeta$ transformation (4·1) is used. We denote by $\mathfrak{Z}_z$ this new cut $z$ plane and denote by $\mathfrak{Z}'_z$ the domain obtained by removing from $\mathfrak{Z}_z$ the domain $\mathbf{L}'_z$. Then we can easily deduce from the nature of the domains in figures 5 and 6, that $\mathfrak{Z}'_\zeta$ is a domain in the $\zeta$ plane which is symmetrical about the real $\zeta$ axis. If we examine appropriate Riemann surfaces in the $z$ plane, we can show that the Airy-type expansions derived for the Ferrers functions $\mathrm{P}_n^{-m}(z)$ and $\mathrm{Q}_n^{-m}(z)$, by the application of theorem B of Olver (1954*b*), are valid uniformly with respect to $z$, as $u\to\infty$, for all values of $z$ in the cut $z$ plane, except for $z$ in $\mathbf{L}'_z$. Thus the expansions for the Ferrers functions are valid in a domain larger than the domain of validity of the expansions for the functions $P_n^{-m}(z)$, $Q_n^{-m}(z)$.

*Formulae for the functions $E_s(\zeta)$ and $F_s(\zeta)$*

These functions are defined by the iteration formulae (4·12). The integrations in (4·12) are difficult to perform but we can obtain explicit expressions for these functions in terms of the coefficients $T_s(\xi)$ derived in (3·9) and (3·21), by a method used by Olver (1954*c*, (6·6)) in obtaining similar functions in the asymptotic expansions of Bessel functions. Hence we derive

$$E_s(\zeta) = \sum_{r=0}^{2s} b_r \zeta^{-\frac{3}{2}s} T_{2s-r}(\xi), \quad \zeta^{\frac{1}{2}} F_s(\zeta) = -\sum_{r=0}^{2s+1} a_r \zeta^{-\frac{3}{2}r} T_{2s-r+1}(\xi), \tag{4·24}$$

where $a_r$, $b_r$ are certain constants which appear in the asymptotic expansions for the functions Ai$(\zeta)$ and Ai$'(\zeta)$ for large values of $\zeta$ (Olver 1954*c*, (6·3)).

*Expansions for the derivatives*

We may differentiate the expansion (4·19) term by term. If we write

$$\phi(\zeta) = \left(\frac{4\zeta}{z^2-\beta^2}\right)^{\frac{1}{4}} = \left(-\frac{2}{z^2-1}\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)^{\frac{1}{2}}, \tag{4·25}$$

we obtain

$$\mathrm{e}^{-\frac{1}{2}m\pi\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}z}P_n^{-m}(z) \sim -\left\{\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}\right\}^{\frac{1}{2}} \psi(z) \left\{\frac{\mathrm{Ai}(u^{\frac{2}{3}}\zeta)}{u^{\frac{1}{3}}} \sum_{s=0}^{\infty} \frac{G_s(\zeta)}{u^{2s}} + u^{\frac{1}{3}}\,\mathrm{Ai}'(u^{\frac{2}{3}}\zeta) \sum_{s=0}^{\infty} \frac{H_s(\zeta)}{u^{2s}}\right\}, \tag{4·26}$$

where
$$\psi(\zeta) = \frac{2}{(z^2-1)\,\phi(\zeta)}, \tag{4·27}$$

$$G_s(\zeta) = \chi(\zeta)\,E_s(\zeta) + E_s'(\zeta) + \zeta F_s(\zeta), \quad H_s(\zeta) = E_s(\zeta) + \chi(\zeta)\,F_{s-1}(\zeta) + F_{s-1}'(\zeta), \tag{4·28}$$

and
$$\chi(\zeta) \equiv \frac{\phi'(\zeta)}{\phi(\zeta)} = \frac{4+z(z^2-1)\,\{\phi(\zeta)\}^6}{16\zeta}.$$

Similar expansions can be derived for the functions $Q_n^{-m}(z)$, $\mathrm{P}_n^{-m}(x)$, $\mathrm{Q}_n^{-m}(x)$. The functions $\phi(\zeta)$, $\psi(\zeta)$, $\chi(\zeta)$, $G_s(\zeta)$ and $H_s(\zeta)$ are regular in the same regions as $z(\zeta)$, and the expansion (4·26) is uniformly valid, as $u\to\infty$, for $z$ lying in the same domain as in (4·19). Finally, from (3·32) and (3·34) we have

$$\zeta^{-\frac{1}{2}}G_s(\zeta) = -\sum_{r=0}^{2s+1} b_r \zeta^{-\frac{3}{2}r} T_{2s-r+1}^{(1)}(\zeta), \quad H_s(\zeta) = \sum_{r=0}^{2s} a_r \zeta^{-\frac{3}{2}r} T_{2s-r}^{(1)}(\zeta).$$

## 5. Expansions for the Legendre functions in terms of Bessel functions

The expansions obtained for $P_n^{-m}(z)$ and $Q_n^{-m}(z)$ in § 4 are uniformly valid as $u\to\infty$ with respect to $z$ in the $z$ plane cut from $+1$ to $-\infty$ except for $z$ lying either in the domain $\mathbf{L}_z'$ encircling $z=-1$ or in the strip $|\mathscr{R}z|<\beta+\delta_1$, $0\geqslant\mathscr{I}z>-\delta_1$ $(\delta_1>0)$. In this present section we apply theorem E of the preceding paper (Thorne 1957*b*; which we refer to as II), to obtain expansions for $P_n^{-m}(z)$ and $Q_n^{-m}(z)$ which are valid both in the regions of the $z$ plane in which the Airy-type expansions of § 4 are valid, and also in the strip mentioned above. These expansions are in terms of modified Bessel functions of large order.

Before proceeding to compare the Legendre equation with the Bessel equation II, (1·4), we first show that it is necessary to use here the parameters $u$, $\alpha$ introduced in (1·3). In II, the transformation II, (2·3), was applied to the equation II, (1·1), to bring it into the form II, (3·1), in which equation the independent variable is $t$. It was then proved in II, § 5, that the solutions of II, (3·1), have asymptotic expansions in terms of Bessel functions which are uniformly valid with respect to $t$, as $u\to\infty$, in unbounded regions of the complex $t$ plane if and only if the corresponding Airy-type expansions for solutions of II, (3·1), are uniformly valid as $|\zeta|\to\infty$ in the sector $|\arg(-\zeta)|<\frac{2}{3}\pi$ in the $\zeta$ plane, where $\zeta$ is the independent variable which appears in the Airy functions (see II, (2·7) and (2·9)).

In the case of the Legendre functions we have shown in § 4 that the point $|z|=\infty$ corresponds to the point $\zeta=\infty\,\mathrm{e}^{\frac{2}{3}\pi i}$, and it was proved in the earlier paper (I, § 5) that the Airy-type expansions for the Legendre functions are uniformly valid with respect to $z$ for $|z|\to\infty$ and thus for $\zeta\to\infty\,\mathrm{e}^{\frac{2}{3}\pi i}$ only if we put the Legendre equation (1·1) in the form (1·3) with the parameters $u=n+\frac{1}{2}$, $\alpha=m/u$. Consequently we deduce from II, § 5, that it is necessary to use the parameters $u$, $\alpha$ in order to obtain Bessel-type expansions for $P_n^{-m}(z)$ and $Q_n^{-m}(z)$ valid uniformly for unbounded $|z|$.

Since the coefficient of $u^2w$ in (1·3) is $(z-1)^{-2}\{\frac{1}{4}\alpha^2+(z-1)\,O(1)\}$ as $z\to 1$, the parameter $\alpha$ which we have to choose in II, (3·1), is identical with that given in (1·3), namely, $\alpha=m/u$ (see II, § 2). Finally, comparing figure 2 of II with figure 5 in this present paper, we see that if we combine the $z$-$\rho$ transformation in (4·1) and the $\rho$-$t$ transformation in II, (2·3), and (2·8) the domain $\mathbf{I}_t'$, say, corresponding to $\mathbf{I}_z'$ in the $z$ plane (§ 4), will lie wholly with $\mathscr{R}t\leqslant 0$.

Since it is more convenient to consider the half-plane $\mathscr{R}t \geqslant 0$, we change the $\rho$-$t$ transformation of II, (2·3), and set

$$\int_{\beta}^{z} \frac{\sqrt{(z^2-\beta^2)}}{z^2-1}\,\mathrm{d}z = -\rho = \int_{\mathrm{i}\alpha}^{t}\left(1+\frac{\alpha^2}{t^2}\right)^{\frac{1}{2}}\mathrm{d}t; \tag{5·1}$$

that is,

$$\left.\begin{aligned}\alpha\tanh^{-1}\left\{\frac{\sqrt{(z^2-\beta^2)}}{\alpha z}\right\}-\tanh^{-1}\left\{\frac{\sqrt{(z^2-\beta^2)}}{z}\right\} = \rho &= \alpha\ln\frac{\mathrm{i}\{\alpha+\sqrt{(t^2+\alpha^2)}\}}{t}-\sqrt{(t^2+\alpha^2)}\\ &= \alpha\cosh^{-1}\frac{\mathrm{i}\alpha}{t}-\sqrt{(t^2+\alpha^2)}.\end{aligned}\right\} \tag{5·2}$$

As in (4·1) we have $\rho = -\xi+\frac{1}{2}\mathrm{i}\alpha\pi$; this differs from II, (4·8). Under the transformation $Y = wz'^{-\frac{1}{2}}$, $z' \equiv \mathrm{d}z/\mathrm{d}t$, the Legendre equation (1·3) becomes

$$\frac{\mathrm{d}^2Y}{\mathrm{d}t^2} = \left\{u^2\left(1+\frac{\alpha^2}{t^2}\right)-\frac{1}{4t^2}+g(t)\right\}Y, \tag{5·3}$$

where
$$g(t) = \frac{(t^2+\alpha^2)\,(z^2-1)\,\{z^2(4\alpha^2-1)+(1-\alpha^4)\}}{4t^2(z^2-\beta^2)^3}+\frac{t^2-4\alpha^2}{4(t^2+\alpha^2)^2}. \tag{5·4}$$

This value of $g(t)$ is calculated from (4·3) and II, (2·12).

*The domain* $\mathbf{I}'_t$

Using the logarithmic form of (5·2) we deduce that as $z \to 1$,

$$z-1 \sim \tfrac{1}{2}p^2\exp\,(2\alpha^{-1}\xi) \sim \frac{p^2\mathrm{e}^2t^2}{8\alpha^2}\{1+O(t^2)\}, \tag{5·5}$$

(compare (3·8)), where $p$ is given in (2·17). Using the logarithmic form of (5·2) again, we find that as $|z| \to \infty$

$$z \sim \tfrac{1}{2}r\beta\,\mathrm{e}^{\xi} \sim C\,\mathrm{e}^{t}, \tag{5·6}$$

where $\ln C = \ln\frac{1}{2}\beta+\ln r$ and $\ln r = \alpha\ln\beta^{-1}(1+\alpha)$. In §4 we have described the domain $\mathbf{I}'_\rho$ in the $\rho$ plane corresponding to the domain $\mathbf{I}'_z$. From the transformation (5·2) we deduce that $\mathbf{I}'_t$ is a domain consisting of a strip which lies wholly with $\mathscr{R}t \geqslant 0$, and which is symmetrical about the real $t$ axis; $\mathbf{I}_{1t}$ and $\mathbf{I}'_{2t}$ lie in $\mathscr{I}t \geqslant 0$ and $\mathbf{I}_{4t}$ and $\mathbf{I}'_{3t}$ are the reflexions of $\mathbf{I}_{1t}$ and $\mathbf{I}'_{2t}$, in that order, in the real $t$ axis. As $|t| \to \infty$ the lines $Q_tF_t$ and $C_tG_t$ are asymptotic to the lines $\mathscr{I}t = \pi$ and $\mathscr{I}t = \frac{1}{2}\pi$ respectively. The points $A_z$ $(z = 1)$, $H_z$ $(z = +\infty)$, $B_z$ $(z = \beta+\mathrm{i}0)$, $B_{1z}$ $(z = \beta-\mathrm{i}0)$ are transformed into the points $t = 0$, $t = \infty$, $t = \mathrm{i}\alpha$, $t = -\mathrm{i}\alpha$, respectively. The points $C_z$ $(z = 0)$, $D_z$ $(z = -\beta+\mathrm{i}0)$, $Q_z$ $(z = \beta\cosh t_1\,\mathrm{e}^{\pi\mathrm{i}})$, $P_z$ $(z = \beta\cosh t_1)$ become the points $t = \mathrm{i}p_1$, $t = \mathrm{i}p_2$, $t = \mathrm{i}p_3$, $t = \alpha\times 0{\cdot}66274\ldots$ respectively, where $p_1$, $p_2$, $p_3$ are the solutions of the equations

$$\left.\begin{aligned}\tfrac{1}{2}(\pi-\alpha) &= \sqrt{(p_1^2-\alpha^2)}-\alpha\,|\cos^{-1}(\alpha/p_1)\,|,\\ \pi(1-\alpha) &= \sqrt{(p_2^2-\alpha^2)}-\alpha\,|\cos^{-1}(\alpha/p_2)\,|,\\ \pi(1-\tfrac{1}{2}\alpha) &= \sqrt{(p_3^2-\alpha^2)}-\alpha\,|\cos^{-1}(\alpha/p_3)\,|.\end{aligned}\right\} \tag{5·7}$$

The constant $\alpha$ is fixed in the range $0<\alpha<1$, but if we let $\alpha \to 0$, then $p_1 \to \frac{1}{2}\pi-0$, $p_2 \to \pi-0$, $p_3 \to \pi-0$ and if $\alpha \to 1$, then $p_1 \to 1+0$, $p_2 \to 1+0$, $p_3 \to \sqrt{(t_3^2+1)} = 2{\cdot}2168\ldots$, where $t_3 = 2{\cdot}0288\ldots$ is that solution of the equation $\tan t_3 = -t_3$ for which $\frac{1}{2}\pi < t_3 < \pi$.

The domain $\mathbf{L}'_{2z}$ becomes a strip $\mathbf{L}'_{2t}$ in $\mathscr{R}t<0$, $\mathscr{I}t>0$, and $\mathbf{L}'_{3z}$ becomes a strip $\mathbf{L}'_{3t}$ in $\mathscr{R}t<0$, $\mathscr{I}t<0$. The Bessel-type expansions derived in this section are not valid in $\mathbf{L}'_{2t}$ and $\mathbf{L}'_{3t}$. The domain $\mathbf{L}_z$ is transformed into the domain $\mathfrak{R}'_t$, described in II, § 2, surrounding $t=0$. If we consider a domain $\mathbf{J}'_z$ (as in § 4), identical with $\mathbf{I}'_z$ and reached by a single counter-clockwise rotation around $z=1$ crossing $A_zB_zC_zD_z$, then $\mathbf{J}'_t$ is the reflexion of $\mathbf{I}'_t$ in the imaginary $t$ axis. Let $B'_{1z}$, $C'_{1z}$, $D'_{1z}$ be the points in $\mathbf{J}'_z$ corresponding to $B_{1z}C_{1z}D_{1z}$ in $\mathbf{I}'_z$. As far as the $z$-$t$ transformation is concerned we may now eliminate the cut $D_{1t}C_{1t}B_{1t}A_tB'_tC'_tD'_t$ in the $t$ plane, and this corresponds to joining the lower side of the cut $-\beta<z<1$ in $\mathbf{I}'_z$ (namely, $A_zB_{1z}C_{1z}D_{1z}$) with the upper side of this cut in $\mathbf{J}'_z$ (namely, $A_zB'_zC'_zD'_z$). Let $\mathbf{T}_t=\mathbf{I}'_t+\mathbf{J}'_t$, where $\mathbf{T}_t$ is cut along the lines $D_tQ_t$ and $D_{1t}Q_{1t}$, given by $z=\mathrm{i}t''$, $t''$ real and $p_2<|t''|<p_3$. Then $\mathbf{T}_t$ is an open simply-connected domain in which $t=0$, $t=\pm\mathrm{i}\alpha$ are interior points. Further, $\mathbf{T}_t$ extends to $t=\pm\infty$ and the distance between the boundaries of $\mathbf{T}_t$ does not tend to zero as $|t|\to\infty$ in $\mathbf{T}_t$. Then $z\equiv z(t)$ is a regular function of $t$ throughout $\mathbf{T}_t$.

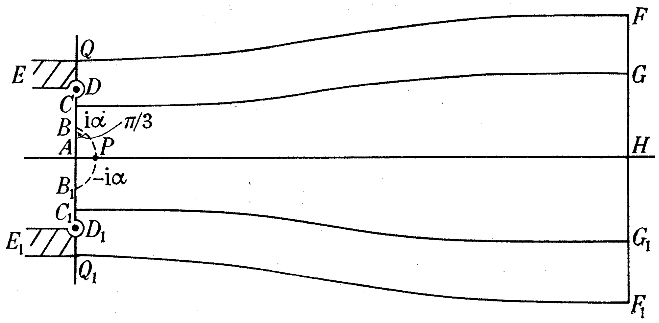


FIGURE 7. $t$ plane.

From (5·4), (5·5) and (5·6) we conclude that $g(t)=O(|t|^{-2})$ as $|t|\to\infty$ in $\mathbf{T}_t$, and $g(t)$ is a regular even function of $t$ throughout $\mathbf{T}_t$ except possibly at the points $t=\pm\mathrm{i}\alpha$. However, from (5·2) we conclude that as $z\to\beta+\mathrm{i}0$, $t\to\mathrm{i}\alpha$ and

$$\frac{\surd(z^2-\beta^2)}{z}=\beta^{-\frac{2}{3}}T(1+a_1T^2+O(T^4)), \tag{5·8}$$

where $T=\surd(t^2+\alpha^2)$ and $a_1=\frac{1}{5}\alpha^{-2}(-2\beta^{-\frac{4}{3}}+1+\beta^{\frac{2}{3}})$. Substituting into (5·4), we can show after some calculation that $g(t)$ is regular at $t=\pm\mathrm{i}\alpha$.

The change of the limits of integration from $t=-\mathrm{i}\alpha$ in II, (2·3), to $t=\mathrm{i}\alpha$ in (5·1) does not affect the application of theorem E. Since $g(t)$ is an even function of $t$, the function $B_s(t)$ in II, (3·5), is an odd function of $t$ and $A_s(t)$ in II, (3·6), is an even function of $t$. Hence, replacing the lower limit of the integral in II, (3·6), does not alter the values of $A_s(t)$ and $B_s(t)$ for $s\geqslant 0$.

We have now shown that all the preliminary conditions for the application of theorem E to solutions of the equation (5·3) in $\mathbf{T}_t$ are satisfied. We take $\mathbf{T}_t$ to correspond to the domain $\mathbf{D}$ of theorem E and define a domain $\mathbf{T}'_t$ to be the domain obtained after removing from $\mathbf{T}_t$ the two strips

$$|\mathscr{R}t|<\delta,\quad p_2-\delta<|\mathscr{I}t|\leqslant p_3. \tag{5·9}$$

We can take $\mathbf{T}'_t$ to correspond to the domain $\mathbf{D}'$ of theorem E since all the conditions of theorem E are satisfied in domains reached by crossing the lines $Q_tF_t$, $F_{1t}Q_{1t}$, $Q'_tF'_t$, $F'_{1t}Q'_{1t}$; we used a similar argument when discussing the domain $\mathbf{Z}'_\xi$ in § 3. Then the domains $\mathbf{D}_0$

and $\mathbf{D}_1$ are identical with $\mathbf{T}'_t$. The two strips in (5·9) correspond to a strip around the boundary of $\mathbf{L}'_z$.

Applying theorem E and using the relations (2·7) and (2·8), we conclude that

$$P_n^{-m}(z) \sim E_u \left(\frac{t^2+\alpha^2}{z^2-\beta^2}\right)^{\frac{1}{4}} \left\{ I_m(ut) \sum_{s=0}^{\infty} \frac{A_s(t)}{u^{2s}} + \frac{I'_m(ut)}{u} \sum_{s=0}^{\infty} \frac{B_s(t)}{u^{2s}} \right\}, \tag{5·10}$$

$$Q_n^{-m}(z) \sim \mathrm{e}^{-m\pi\mathrm{i}} F_u \left(\frac{t^2+\alpha^2}{z^2-\beta^2}\right)^{\frac{1}{4}} \left\{ K_m(ut) \sum_{s=0}^{\infty} \frac{A_s(t)}{u^{2s}} + \frac{K'_m(ut)}{u} \sum_{s=0}^{\infty} \frac{B_s(t)}{u^{2s}} \right\}, \tag{5·11}$$

where $A_s(t)$ and $B_s(t)$ are given by the relations

$$A_0(t) = 1, \quad B_s(t) = \frac{1}{2}\left(1+\frac{\alpha^2}{t^2}\right)^{-\frac{1}{2}} \int_{\mathrm{i}\alpha}^{t} \left\{ g(v)\, A_s(v) - \frac{1}{v} A'_s(v) - A''_s(v) \right\} \left(1+\frac{\alpha^2}{v^2}\right)^{-\frac{1}{2}} \mathrm{d}v, \tag{5·12}$$

$$A_{s+1}(t) = -\tfrac{1}{2} B'_s(t) + \frac{1}{2t} B_s(t) + \frac{1}{2}\int_0^t g(v)\, B_s(v)\, \mathrm{d}v + a_{s+1} \quad (s \geqslant 0), \tag{5·13}$$

with $g(t)$ as given in (5·4).

We suppose that the integration constants $a_s$ in (5·13) are real. The coefficients $E_u$ and $F_u$ in (5·10) and (5·11) are now derived in a manner similar to that used in § 4 in the derivation of $C_u$ and $D_u$. By examining the expansions (5·10) and (5·11) for a point $R_t$, say, on $A_t H_t$, and taking the integration in (5·12) along the lines $B_t A_t$, $A_t H_t$ we can show that $E_u$ and $F_u$ are real functions of $u$. Now taking $R_t$ to lie on $B_t A_t$ we can conclude that $E_u = F_u$. Then letting $R_t \to A_t$ we derive

$$\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} \left(\frac{p\mathrm{e}}{2m}\right)^{-m} \sim E_u \left(a_0 - \frac{b_0}{u} + \frac{a_1}{u^2} - \frac{b_1}{u^3} + \ldots\right), \tag{5·14}$$

$$\left(\frac{p\mathrm{e}}{2m}\right)^{m} \sim F_u \left(a_0 + \frac{b_0}{u} + \frac{a_1}{u^2} + \frac{b_1}{u^3} + \ldots\right), \tag{5·15}$$

where
$$a_s = A_s(0), \quad b_s = \alpha \lim_{t \to 0} \{t^{-1} B_s(t)\}. \tag{5·16}$$

Hence, using the methods of § 4, we deduce that

$$P_n^{-m}(z) \sim \left\{\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}\right\}^{\frac{1}{2}} \left(\frac{t^2+\alpha^2}{z^2-\beta^2}\right)^{\frac{1}{4}} \left\{ I_m(ut) \sum_{s=0}^{\infty} \frac{A_s(t)}{u^{2s}} + \frac{I'_m(ut)}{u} \sum_{s=0}^{\infty} \frac{B_s(t)}{u^{2s}} \right\}, \tag{5·17}$$

$$Q_n^{-m}(z) \sim \mathrm{e}^{-m\pi\mathrm{i}} \left\{\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}\right\}^{\frac{1}{2}} \left(\frac{t^2+\alpha^2}{z^2-\beta^2}\right)^{\frac{1}{4}} \left\{ K_m(ut) \sum_{s=0}^{\infty} \frac{A_s(t)}{u^{2s}} + \frac{K'_m(ut)}{u} \sum_{s=0}^{\infty} \frac{B_s(t)}{u^{2s}} \right\}, \tag{5·18}$$

where
$$\sum_{s=0}^{\infty} \left\{\frac{a_s}{u^{2s}} + \frac{b_s}{u^{2s+1}}\right\} \sim \left(\frac{p\mathrm{e}}{2m}\right)^{m} \left\{\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}\right\}^{\frac{1}{2}}, \tag{5·19}$$

and $a_s$ and $b_s$ are given in (5·16). The $z$-$t$ relation is given in (5·2).

These expansions are uniformly valid with respect to $z$ as $u \to \infty$ throughout the $z$ plane cut from $+1$ to $-\infty$, except for the pear-shaped domain $\mathbf{L}'_z$ surrounding the point $z = -1$. To obtain expansions valid inside $\mathbf{L}'_z$ we may use the continuation formulae (2·12) and (2·13). From the known behaviour of the Bessel functions $I_m$ and $K_m$ over the $t$ plane (see Olver 1954*c*), we may deduce properties of the Legendre functions over the $z$ plane as $u \to \infty$; these properties have been given in § 4. If $m$ is a positive integer we may deduce the expansion for $P_n^m(z)$ immediately from (5·17).

*Expansions for* $\mathrm{P}_n^{-m}(x)$, $\mathrm{Q}_n^{-m}(x)$

If $t\pm$ corresponds to the points $z = x\pm \mathrm{i}0$, where $x$ is real and $-\beta < x < 1$, then $t\pm = \mathrm{e}^{\pm\frac{1}{2}\pi\mathrm{i}}\,\tau$ where $\tau$ is real. Then we can write

$$A_s(t\pm) = \mathscr{A}_s(\tau), \quad \pm \mathrm{i}B_s(t\pm) = \mathscr{B}_s(\tau),$$

$$I_m(ut\pm) = \mathrm{e}^{\pm\frac{1}{2}m\pi\mathrm{i}}\,J_m(u\tau), \quad K_m(ut\pm) = \tfrac{1}{2}\pi\operatorname{cosec} m\pi\{\mathrm{e}^{\mp\frac{1}{2}m\pi\mathrm{i}}\,J_{-m}(u\tau) - \mathrm{e}^{\pm\frac{1}{2}m\pi\mathrm{i}}\,J_m(u\tau)\},$$

$$\mathrm{e}^{\frac{1}{2}m\pi\mathrm{i}}\,K_m(ut+) + \mathrm{e}^{-\frac{1}{2}m\pi\mathrm{i}}\,K_m(ut-) = -\pi Y_m(u\tau).$$

Hence we deduce for these values of $x$,

$$\mathrm{P}_n^{-m}(x) \sim \left\{\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}\right\}^{\frac{1}{2}} \left(\frac{\alpha^2-\tau^2}{z^2-\beta^2}\right)^{\frac{1}{4}} \left\{J_m(u\tau)\sum_{s=0}^{\infty}\frac{\mathscr{A}_s(\tau)}{u^{2s}} + \frac{J'_m(u\tau)}{u}\sum_{s=0}^{\infty}\frac{\mathscr{B}_s(\tau)}{u^{2s}}\right\}, \tag{5·20}$$

$$\mathrm{Q}_n^{-m}(x) \sim -\frac{\pi}{2}\left\{\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}\right\}^{\frac{1}{2}} \left(\frac{\alpha^2-\tau^2}{z^2-\beta^2}\right)^{\frac{1}{4}} \left\{Y_m(u\tau)\sum_{s=0}^{\infty}\frac{\mathscr{A}_s(\tau)}{u^{2s}} + \frac{Y'_m(u\tau)}{u}\sum_{s=0}^{\infty}\frac{\mathscr{B}_s(\tau)}{u^{2s}}\right\}, \tag{5·21}$$

where $$\alpha\cosh^{-1}\left\{\frac{\alpha x}{\beta\surd(1-x^2)}\right\} - \cosh^{-1}\frac{x}{\beta} = \alpha\cosh^{-1}\frac{\alpha}{\tau} - \surd(\alpha^2-\tau^2).$$

*Determination of the functions* $A_s(t)$ *and* $B_s(t)$

We can obtain $B_0(t)$ by direct integration from (5·12). Thus, noting that the lower limit in the integrals gives a zero contribution since $g(t)$ is bounded at $t = \mathrm{i}\alpha$, we derive

$$B_0(t) = \frac{t^2(2\alpha^2-3t^2)}{24(t^2+\alpha^2)} - \frac{tz\{z^2(2\alpha^2-3)+3(1-\alpha^4)\}}{24\beta^2(z^2-\beta^2)^{\frac{3}{2}}\,(t^2+\alpha^2)^{\frac{1}{2}}}. \tag{5·22}$$

The derivation of the other functions by this means is not practicable, but we can obtain expansions for the functions in a manner similar to that used in § 4, and Olver (1954*c*, § 6). Let $z$ be a fixed point in the half-plane $\mathscr{R}z \geqslant 0$ not lying near $z = \beta$, or near the real $z$ axis. Then the expansions (5·17), (3·11), and II, (4·7), are all valid. By substitution we deduce that

$$\sum_{s=0}^{\infty}\frac{T_s(\xi)}{u^s} \sim \sum_{s=0}^{\infty}\frac{U_s(\xi/\alpha)}{m^s}\sum_{s=0}^{\infty}\frac{A_s(t)}{u^{2s}} + \sum_{s=0}^{\infty}\frac{V_s(\xi/\alpha)}{m^s}\sum_{s=0}^{\infty}\left(1+\frac{\alpha^2}{t^2}\right)^{\frac{1}{2}}\frac{B_s(t)}{u^{2s+1}}.$$

Equating coefficients of $u^{-s}$ and using the identity

$$U_{2s}V_0 - U_{2s-1}V_1 + \ldots + U_0V_{2s} = 1 \quad (s \geqslant 1),$$

from Olver (1954*c*, (2·24)) we obtain by direct addition

$$\left.\begin{aligned} A_s(t) &= \sum_{r=0}^{2s}(-\alpha)^{-r}\,V_r(\xi/\alpha)\,T_{2s-r}(\xi), \\ \left(1+\frac{\alpha^2}{t^2}\right)^{\frac{1}{2}}B_s(t) &= \sum_{r=0}^{2s+1}(-\alpha)^{-r}\,U_r(\xi/\alpha)\,T_{2s+1-r}(\xi). \end{aligned}\right\} \tag{5·23}$$

We deduce from the principle of analytic continuation that the expressions (5·23) hold throughout the region of validity of the expansions (5·17) and (5·18).

*Expansions for the derivatives*

If we write
$$\phi_1(t) = \left(\frac{t^2+\alpha^2}{z^2-\beta^2}\right)^{\frac{1}{4}} = \left(\frac{t}{z^2-1}\frac{dz}{dt}\right)^{\frac{1}{2}}, \tag{5·24}$$

then
$$\frac{d}{dz}P_n^{-m}(z) \sim \left\{\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}\right\}^{\frac{1}{2}} \psi_1(t)\left\{\frac{I_m(ut)}{t}\sum_{s=0}^{\infty}\frac{C_s(t)}{u^{2s}} + uI_m'(ut)\sum_{s=0}^{\infty}\frac{D_s(t)}{u^{2s}}\right\}, \tag{5·25}$$

where
$$\psi_1(t) = \frac{t}{(z^2-1)\,\phi_1(t)}, \tag{5·26}$$

$$\left.\begin{aligned} C_s(t) &= tA_s'(t) + t\chi_1(t)\,A_s(t) + t\left(1+\frac{\alpha^2}{t^2}\right)B_s(t), \\ D_s(t) &= A_s(t) + B_{s-1}'(t) - \frac{1}{t}B_{s-1}(t) + \chi_1(t)\,B_{s-1}(t), \end{aligned}\right\} \tag{5·27}$$

where
$$\chi_1(t) = \frac{\phi_1'(t)}{\phi_1(t)} = \frac{1-2z(z^2-1)\{\phi_1(t)\}^6}{4(t^2+\alpha^2)}.$$

Similar expansions can be derived for the other Legendre functions. The functions $\phi_1(t)$, $\chi_1(t)$, $\psi_1(t)$, $C_s(t)$ and $D_s(t)$ are analytic in the same regions as $z \equiv z(t)$, and the expansion (5·25) is uniformly valid, as $u \to \infty$, for $z$ lying in the same domain as in (5·17).

## 6. Previous results and conclusion

It was mentioned in §1 that no expansions have been developed previously for the Legendre functions of large degree $n$ and order $m$. Numerous expansions have been given for the Legendre functions of fixed order ($m$) and large degree ($n$). All these expansions can be obtained from certain expansions obtained by Olver (1954*a*). The functions

$$(\sinh t)^{\frac{1}{2}}P_n^{-m}(\cosh t), \quad (\sinh t)^{\frac{1}{2}}Q_n^{-m}(\cosh t)$$

are the solutions of the differential equation

$$\frac{d^2y}{dt^2} = \left\{u^2 + \frac{m^2}{t^2} - \frac{1}{4t^2} + (m^2-\tfrac{1}{4})\left(\frac{1}{\sinh^2 t} - \frac{1}{t^2}\right)\right\}y, \tag{6·1}$$

where $u = n+\frac{1}{2}$. This equation is of the type II, (7·1), and the expansions for the Legendre functions are in terms of the Bessel functions $I_m(ut)$ and $K_m(ut)$. The leading term of the expansion corresponding to $P_n^{-m}(x)$ has been given previously by Szegö (1939, (8·21·17)). It is not difficult to see the relation between these expansions and the Bessel-type expansions of §5. If we let $\alpha \to 0$, then the $z$-$t$ transformation (5·2) becomes $t \sim \cosh^{-1} z$, and the leading term of (5·20) is the same as the term given by Szegö.

The Legendre functions of large degree and order have properties similar to certain hypergeometric functions $G_\nu \equiv \tau^{\frac{1}{2}\nu}F(a_\nu, b_\nu; \nu+1, \tau)$ considered by Cherry (1947, §2, 1949, 1950*a*, *b*). In these functions $\beta$ is fixed, and $|\nu| \to \infty$ with $2a_\nu = \nu - \beta + \surd\{\nu^2(1+2\beta)+\beta^2\}$, $2b_\nu = \nu - \beta - \surd\{\nu^2(1+2\beta)+\beta^2\}$; these functions appear in gas-flow theory. The equation satisfied by $G_\nu$ has, for large values of $\nu$, a form similar to (1·3) for large $u$. Using the method of steepest descents, Cherry (1947, 1949) has derived asymptotic expansion for $G_\nu$ in terms of exponential functions; these may be compared with the expansions in §3. Cherry (1950*a*, *b*) has also given expansions which are valid at the transition point of the differential equation satisfied by $G_\nu$, corresponding to the point $z = \beta$ in (1·5); these expansions are in terms of Bessel functions of large order and are derived by the method described in II, § 7. We note that if Cherry's method were applied to the Legendre equation, the resulting Bessel-type expansions would have the same domain of validity as the expansions of § 4.

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